Long-Time Asymptotics of Solutions to the Cauchy Problem for the Defocusing Non-Linear Schrödinger Equation with Finite-Density Initial Data. II. Dark Solitons on Continua

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21 October 2002

Abstract

For Lax-pair isospectral deformations whose associated spectrum, for given initial data, consists of the disjoint union of a finitely denumerable discrete spectrum (solitons) and a continuous spectrum (continuum), the matrix Riemann-Hilbert problem approach is used to derive the leading-order asymptotics as $|t| \to \infty$ $(x/t \sim \mathcal{O}(1))$ of solutions $(u = u(x,t))$ to the Cauchy problem for the defocusing non-linear Schrödinger equation (D$_f$NLSE), $\mathrm{i}\partial_t u + \partial_x^2 u - 2(|u|^2 - 1)u = 0$, with finite-density initial data $u(x,0) =_{x \to \pm\infty} \exp(\frac{\mathrm{i}(1 \mp 1)\theta}{2})(1 + o(1))$, $\theta \in [0, 2\pi)$. The D$_f$NLSE dark soliton position shifts in the presence of the continuum are also obtained.

2000 Mathematics Subject Classification. (Primary) 35Q15, 37K40, 35Q55, 37K15: (Secondary) 30E20, 30E25, 81U40

Abbreviated Title. Asymptotics of the D$_f$NLSE Dark Solitons on Continua

Key Words. Asymptotics, direct and inverse scattering, reflection coefficient, Riemann-Hilbert problems, singular integral equations

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1 Introduction

In direct detection systems making use of polarisation-preserving single-mode (PPSM) optical fibres, return-to-zero bright soliton (strictly speaking, soliton-like) pulses, which propagate in the anomalous group velocity dispersion (GVD) regime (wavelengths $>1.3\mu$m in standard telecommunications fibres), have been shown to be effective toward the partial resolution of the deleterious problem of performance degradation caused by, for example, dispersive pulse spreading [1]. For coherent communications systems, non-return-to-zero dark soliton pulses, which propagate in the normal GVD regime (wavelengths $<1.3\mu$m) and consist of a rapid dip in the intensity of a broad pulse of a continuous wave background, offer an analogous benefit [2, 3, 4].

A model for dark soliton pulse propagation in PPSM optical fibres in the picosecond time scale, which describes the slowly varying amplitude of the complex field envelope, $u=u(x,t)$, in normalised and dimensionless form, is the Cauchy problem for the defocusing non-linear Schrödinger equation (D_fNLSE) with finite-density, or non-vanishing, initial data [1, 2, 3, 4],

$$\begin{gathered}\mathrm{i}\partial_t u+\partial_x^2 u-2(|u|^2-1)u=0, \qquad (x,t)\in\mathbb{R}\times\mathbb{R},\\ u(x,0):=u_o(x)\underset{x\to\pm\infty}{=}\exp(\tfrac{\mathrm{i}(1\mp1)\theta}{2})(1+o(1)),\end{gathered}\tag{1}$$

where $u_o(x)\in\mathbf{C}^\infty(\mathbb{R})$, $\theta\in[0,2\pi)$ (see Eq. (3)), and $o(1)$ is to be understood in the sense that, $\forall\,(k,l)\in\mathbb{Z}_{\geqslant0}\times\mathbb{Z}_{\geqslant0}$, $|x|^k(\frac{\mathrm{d}}{\mathrm{d}x})^l(u_o(x)-\exp(\frac{\mathrm{i}(1\mp1)\theta}{2}))=_{x\to\pm\infty}0$. It is shown in [5] that, for initial data satisfying $|x|^k(\frac{\mathrm{d}}{\mathrm{d}x})^l(u_o(x)-\exp(\frac{\mathrm{i}(1\mp1)\theta}{2}))=_{x\to\pm\infty}0$, $(k,l)\in\mathbb{Z}_{\geqslant0}\times\mathbb{Z}_{\geqslant0}$, the closure of the set of soliton, or reflectionless, potentials of the D_fNLSE in the topology of uniform convergence of functions on compact sets of $\mathbb{R}$ remains an invariant set of the model $\forall\,t\in\mathbb{R}$ and not just for $t=0$ (see, also, [6]).

When (temporal) dark solitons are launched sufficiently close together in optical fibres, they interact not only through soliton-soliton interactions, but also through soliton-radiation-tail interactions. Such interactions manifest as a jitter in the arrival times of dark solitons, potentially resulting in their shift outside of some predetermined timing window and giving rise to errors in the detected information [4]. Physically, the optical pulse adjusts its width as it propagates along the optical fibre to evolve into a (multi-) dark soliton pulse/mode, and a part, however small, of the pulse energy is shed in the form of an asymptotically decaying dispersive wavetrain, manifesting as a low-level broadband background radiation (a continuum of linear-like radiative waves/modes). Modulo an $\mathcal{O}(1)$ position shift due to cummulative interactions with other dark solitons and the (dispersive) continuum, the dark soliton pulse/mode maintains its robust/stable properties. From the physical and theoretical point of view, therefore, it is important to understand how the dark solitons and continuum interact, and to be able to derive an explicit functional form for this process, namely, to study the asymptotics as $|t|\to\infty$ $(x/t\sim\mathcal{O}(1))$ of solutions to the Cauchy problem for the D_fNLSE with finite-density initial data having a (not the only one possible) decomposition of the form $u_o(x):=u_{\mathrm{sol}}(x)+u_{\mathrm{rad}}(x)$, where $u_o(x)$ satisfies the conditions stated heretofore, $u_{\mathrm{sol}}(x)$ "generates" the multi- or N-dark soliton component of the solution, and $u_{\mathrm{rad}}(x)$ is the "small" non-dark-soliton part giving rise to the dispersive component of the solution. In this paper, the leading- ($\mathcal{O}(1)$) and next-to-leading-order ($\mathcal{O}(|t|^{-1/2})$) terms of the asymptotic expansion as $|t|\to\infty$ $(x/t\sim\mathcal{O}(1))$ of the solution to the Cauchy problem for the D_fNLSE with finite-density initial data are derived: they represent, respectively, the N-dark soliton component, and the dispersive continuum and non-trivial interaction/overlap of the N-dark solitons with the continuum.

Within the framework of the inverse scattering method (ISM) [7, 8, 9] (see, also, [10]), it is well known that the D_fNLSE is a completely integrable non-linear evolution equation (NLEE) having a representation as an infinite-dimensional Hamiltonian system [11, 12]. Even though the analysis of completely integrable NLEEs with rapidly decaying, e.g., Schwartz class, initial data on $\mathbb{R}$ have received the vast majority of the attention within the ISM framework, there have been a handful of works devoted exclusively to the direct and inverse scattering analysis of completely integrable NLEEs belonging to the ZS-AKNS class with non-vanishing (as $|x|\to\infty$) values of the initial data [13, 14, 15] (see, also, [16, 17]). Other, very interesting classes of finite-density-type initial data for completely integrable NLEEs have also been considered [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32].

Within the ISM framework, the asymptotic analysis of the solution to the Cauchy problem for the D_fNLSE with finite-density initial data is divided into two steps: (1) the analysis of the *solitonless* (pure radiative, or continuous) component of the solution; and (2) the inclusion of the N-dark soliton component via the application of a "dressing" procedure to the solitonless background [33, 34, 35,

36, 37]. The complete details of the asymptotic analysis that constitutes stage (1) of the two-step asymptotic paradigm above, which is quite technical and whose results are essential in order to obtain those of the present paper, can be found in [38]: this paper addresses stage (2) of the above programme via the matrix Riemann-Hilbert problem (RHP) approach [7, 12, 39, 40, 41, 42, 43, 44, 45, 46, 47]. It is important to note that, to the best of the author's knowledge as at the time of the presents, the first to obtain asymptotics of solutions to the Cauchy problem for the D$_f$NLSE with finite-density initial data in the solitonless sector were Its and Ustinov [48, 49].

This paper is organized as follows. In Section 2, the necessary facts from the direct and inverse scattering analysis for the D$_f$NLSE with finite-density initial data are given, the (matrix) RHP analysed asymptotically as $|t| \to \infty$ $(x/t \sim \mathcal{O}(1))$ is stated, and the results of this paper are summarised in Theorems 2.2.1–2.2.4 (and Corollaries 2.2.1 and 2.2.2). In Section 3, an augmented RHP, which is equivalent to the original one stated in Section 2, is formulated, and it is shown that, as $t \to +\infty$, modulo exponentially small terms, the solution of the augmented RHP converges to the solution of an explicitly solvable, model RHP. In Section 4, the model RHP is solved asymptotically as $t \to +\infty$, from which the asymptotics of $u(x,t)$ and $\int_{\pm\infty}^{x}(|u(x',t)|^2-1)\,\mathrm{d}x'$ are derived, and, in Appendix A, the—analogous—asymptotic analysis is succinctly reworked for the case when $t \to -\infty$. In Appendices B and C, respectively, formulae which are necessary in order to obtain the remaining asymptotic results of this paper are presented, and a panoramic view of the matrix RH theory in the L^2-Sobolev space is given [44, 45, 46, 50].

2 The Riemann-Hilbert Problem and Summary of Results

In this section, a synopsis of the direct/inverse spectral analysis for Eq. (1) is given, the matrix RHP studied asymptotically as $|t| \to \infty$ $(x/t \sim \mathcal{O}(1))$ is stated, and the results of this paper are summarised in Theorems 2.2.1–2.2.4. Before doing so, however, it will be convenient to introduce the notation used throughout this work.

<u>Notational Conventions</u>

(1) $\mathrm{I} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is the 2×2 identity matrix, $\mathbf{0} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma_2 = \left(\begin{smallmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{smallmatrix}\right)$, and $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ are the Pauli matrices, $\sigma_+ = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $\sigma_- = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ are, respectively, the raising and lowering matrices, $\mathrm{sgn}(x) := 0$ if $x = 0$ and $x|x|^{-1}$ if $x \neq 0$, and $\mathbb{R}_\pm := \{x;\ \pm x > 0\}$;

(2) for a scalar ω and a 2×2 matrix Υ, $\omega^{\mathrm{ad}(\sigma_3)}\Upsilon := \omega^{\sigma_3}\Upsilon\omega^{-\sigma_3}$;

(3) for each segment of an oriented contour $\mathcal{D}$, according to the given orientation, the "+" side is to the left and the "-" side is to the right as one traverses the contour in the direction of orientation, that is, for a matrix $\mathcal{A}_{ij}(\cdot)$, $i,j \in \{1,2\}$, $(\mathcal{A}_{ij}(\cdot))_\pm$ denote the non-tangential limits $(\mathcal{A}_{ij}(z))_\pm := \lim_{\substack{z' \to z \\ z' \in \pm \text{ side of } \mathcal{D}}} \mathcal{A}_{ij}(z')$;

(4) for a matrix $\mathcal{A}_{ij}(\cdot)$, $i,j \in \{1,2\}$, to have boundary values in the $\mathcal{L}^2$ sense on an oriented contour $\mathcal{D}$, it is meant that $\lim_{\substack{z' \to z \\ z' \in \pm \text{ side of } \mathcal{D}}} \int_{\mathcal{D}} |\mathcal{A}(z') - (\mathcal{A}(z))_\pm|^2\,|\mathrm{d}z| = 0$, where $|\mathcal{A}(\cdot)|$ denotes the Hilbert-Schmidt norm, $|\mathcal{A}(\cdot)| := (\sum_{i,j=1}^{2} \overline{\mathcal{A}_{ij}(\cdot)}\,\mathcal{A}_{ij}(\cdot))^{1/2}$, with $\overline{(\bullet)}$ denoting complex conjugation of $(\bullet)$;

(5) for $1 \leqslant p < \infty$ and $\mathcal{D}$ some point set,

$$\mathcal{L}^p_{\mathrm{M}_2(\mathbb{C})}(\mathcal{D}) := \{f\colon \mathcal{D} \to \mathrm{M}_2(\mathbb{C});\ ||f(\cdot)||_{\mathcal{L}^p_{\mathrm{M}_2(\mathbb{C})}(\mathcal{D})} := (\textstyle\int_{\mathcal{D}} |f(z)|^p\,|\mathrm{d}z|)^{1/p} < \infty\},$$

and, for $p = \infty$,

$$\mathcal{L}^\infty_{\mathrm{M}_2(\mathbb{C})}(\mathcal{D}) := \{g\colon \mathcal{D} \to \mathrm{M}_2(\mathbb{C});\ ||g(\cdot)||_{\mathcal{L}^\infty_{\mathrm{M}_2(\mathbb{C})}(\mathcal{D})} := \max_{i,j \in \{1,2\}} \sup_{z \in \mathcal{D}} |g_{ij}(z)| < \infty\};$$

(6) for D an unbounded domain of $\mathbb{R}$, $\mathcal{S}_{\mathbb{C}}(D)$ (respectively, $\mathcal{S}_{\mathrm{M}_2(\mathbb{C})}(D)$) denotes the Schwartz space on D, namely, the space of all infinitely continuously differentiable (smooth) $\mathbb{C}$-valued (respectively, $\mathrm{M}_2(\mathbb{C})$-valued) functions which together with all their derivatives tend to zero faster than any positive power of $|\bullet|^{-1}$ as $|\bullet| \to \infty$, that is, $\mathcal{S}_{\mathbb{C}}(D) := \mathbf{C}^\infty(D) \cap \{f\colon D \to \mathbb{C};\ ||f(\cdot)||_{k,l} := \sup_{x\in\mathbb{R}} |x^k (\frac{\mathrm{d}}{\mathrm{d}x})^l f(x)| < \infty,\ (k,l) \in \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}\}$ and $\mathcal{S}_{\mathrm{M}_2(\mathbb{C})}(D) := \{F\colon D \to \mathrm{M}_2(\mathbb{C});\ F_{ij}(\cdot) \in \mathbf{C}^\infty(D),\ i,j \in \{1,2\}\} \cap \{G\colon D \to \mathrm{M}_2(\mathbb{C});\ ||G_{ij}(\cdot)||_{k,l} := \max_{i,j\in\{1,2\}} \sup_{x\in\mathbb{R}} |x^k (\frac{\mathrm{d}}{\mathrm{d}x})^l G_{ij}(x)| < \infty,\ (k,l) \in \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}\}$, and $\mathbf{C}^\infty_0(*) := \cap_{k=0}^{\infty} \mathbf{C}^k_0(*)$;

(7) for D an unbounded domain of $\mathbb{R}$, $\mathcal{S}^1_{\mathbb{C}}(D) := \mathcal{S}_{\mathbb{C}}(D) \cap \{h(z);\, ||h(\cdot)||_{\mathcal{L}^\infty(D)} := \sup_{z\in D}|h(z)| < 1\}$;

(8) $||\mathcal{F}(\cdot)||_{\cap_{p\in J}\mathcal{L}^p_{\mathrm{M}_2(\mathbb{C})}(*)} := \sum_{p\in J}||\mathcal{F}(\cdot)||_{\mathcal{L}^p_{\mathrm{M}_2(\mathbb{C})}(*)}$, with $\mathrm{card}(J) < \infty$;

(9) for $(\mu,\widetilde{\nu}) \in \mathbb{R}\times\mathbb{R}$, the function $(\bullet-\mu)^{\mathrm{i}\widetilde{\nu}} \colon \mathbb{C}\setminus(-\infty,\mu) \to \mathbb{C} \colon \bullet \mapsto \mathrm{e}^{\mathrm{i}\widetilde{\nu}\ln(\bullet-\mu)}$, with the branch cut taken along $(-\infty,\mu)$ and the principal branch of the logarithm chosen, $\ln(\bullet-\mu) := \ln|\bullet-\mu| + \mathrm{i}\arg(\bullet-\mu)$, $\arg(\bullet-\mu) \in (-\pi,\pi)$;

(10) a contour, $\mathcal{D}$, say, which is the finite union of piecewise-smooth, simple, closed curves, is said to be *orientable* if its complement, $\mathbb{C}\setminus\mathcal{D}$, can always be divided into two, possibly disconnected, disjoint open sets $\mho^+$ and $\mho^-$, either of which has finitely many components, such that $\mathcal{D}$ admits an orientation so that it can either be viewed as a positively oriented boundary $\mathcal{D}^+$ for $\mho^+$ or as a negatively oriented boundary $\mathcal{D}^-$ for $\mho^-$ [45], i.e., the (possibly disconnected) components of $\mathbb{C}\setminus\mathcal{D}$ can be coloured by $+$ or $-$ in such a way that the $+$ regions do not share boundary with the $-$ regions, except, possibly, at finitely many points [46];

(11) for $\boldsymbol{\gamma}$ a nullhomologous path in a region $\mathfrak{D} \subset \mathbb{C}$, $\mathrm{int}(\boldsymbol{\gamma}) := \{\zeta \in \mathfrak{D}\setminus\boldsymbol{\gamma};\, \mathrm{ind}_{\boldsymbol{\gamma}}(\zeta) := \frac{1}{2\pi\mathrm{i}}\int_{\boldsymbol{\gamma}}\frac{\mathrm{d}\zeta'}{\zeta'-\zeta} \neq 0\}$.

2.1 The RHP for the D_fNLSE

In this subsection, the main results from the direct/inverse scattering analysis of the Cauchy problem for the D_fNLSE are succinctly recapitulated: since the proofs of these results are given in [38], only final results are stated.

Proposition 2.1.1. *The necessary and sufficient condition for the compatibility of the following linear system (Lax-pair), for arbitrary* $\zeta \in \mathbb{C}$,

$$\partial_x\Psi(x,t;\zeta) = \mathcal{U}(x,t;\zeta)\Psi(x,t;\zeta), \qquad \partial_t\Psi(x,t;\zeta) = \mathcal{V}(x,t;\zeta)\Psi(x,t;\zeta), \tag{2}$$

where

$$\mathcal{U}(x,t;\zeta) = -\mathrm{i}\lambda(\zeta)\sigma_3 + \begin{pmatrix} 0 & u \\ \overline{u} & 0 \end{pmatrix},$$

$$\mathcal{V}(x,t;\zeta) = -2\mathrm{i}(\lambda(\zeta))^2\sigma_3 + 2\lambda(\zeta)\begin{pmatrix} 0 & u \\ \overline{u} & 0 \end{pmatrix} - \mathrm{i}\begin{pmatrix} u\overline{u}-1 & \partial_x u \\ \partial_x\overline{u} & u\overline{u}-1 \end{pmatrix}\sigma_3,$$

and $\lambda(\zeta) = \frac{1}{2}(\zeta+\zeta^{-1})$*, with* $\partial_*\zeta = 0$*,* $* \in \{x,t\}$*, is that* $u = u(x,t)$ *satisfies the* D_fNLSE.

One proves Proposition 2.1.1 via the isospectral deformation condition ($\partial_*\zeta = 0$, $* \in \{x,t\}$), and invoking the Frobenius compatibility condition, $\partial_t\partial_x\Psi(x,t;\zeta) = \partial_x\partial_t\Psi(x,t;\zeta) \Rightarrow \partial_t\mathcal{U}(x,t;\zeta) - \partial_x\mathcal{V}(x,t;\zeta) + [\mathcal{U}(x,t;\zeta),\mathcal{V}(x,t;\zeta)] = \mathbf{0}$, $\zeta \in \mathbb{C}$, where $[\mathcal{A},\mathcal{B}] := \mathcal{A}\mathcal{B}-\mathcal{B}\mathcal{A}$ is the matrix commutator (note that $\mathrm{tr}(\mathcal{U}(x,t;\zeta)) = \mathrm{tr}(\mathcal{V}(x,t;\zeta)) = 0$).

Remark 2.1.1. Note that, if $u(x,t)$ is a solution of the D_fNLSE with $\Psi(x,t;\zeta)$ the corresponding solution of system (2), $\Psi(x,t;\zeta)\mathcal{Q}(\zeta)$, with $\mathcal{Q}(\zeta) \in \mathrm{M}_2(\mathbb{C})$, also solves system (2).

The ISM analysis for the D_fNLSE is based on the direct scattering problem for the (self-adjoint) operator (cf. Proposition 2.1.1) $\mathcal{O}^{\mathcal{D}} := \mathrm{i}\sigma_3\partial_x - \left(\begin{smallmatrix} \frac{1}{2}(\zeta+\zeta^{-1}) & \mathrm{i}u_o(x) \\ \mathrm{i}\overline{u_o(x)} & \frac{1}{2}(\zeta+\zeta^{-1}) \end{smallmatrix}\right)$, where $u(x,0) := u_o(x)$ satisfies $u_o(x) =_{x\to\pm\infty} u_o(\pm\infty)(1+o(1))$, with $u_o(\pm\infty) := \exp(\frac{\mathrm{i}(1\mp1)\theta}{2})$, $\theta \in [0,2\pi)$ (see Eq. (3)), $u_o(x) \in \mathbf{C}^\infty(\mathbb{R})$, and $u_o(x) - u_o(\pm\infty) \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}_\pm)$.

Definition 2.1.1. *Let* $u(x,t)$ *be the solution of the* D_fNLSE *with* $u(x,0) := u_o(x) =_{x\to\pm\infty} u_o(\pm\infty)(1+o(1))$*, where* $u_o(\pm\infty) := \exp(\frac{\mathrm{i}(1\mp1)\theta}{2})$*,* $\theta \in [0,2\pi)$ *(see Eq. (3)),* $u_o(x) \in \mathbf{C}^\infty(\mathbb{R})$*, and* $u_o(x) - u_o(\pm\infty) \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}_\pm)$*. Define* $\Psi^\pm(x,0;\zeta)$ *as the (Jost) solutions of the first equation of system* (2)*,* $\mathcal{O}^{\mathcal{D}}\Psi^\pm(x,0;\zeta) = \mathbf{0}$*, with the following asymptotics:*

$$\Psi^\pm(x,0;\zeta) \underset{x\to\pm\infty}{=} \left(\mathrm{e}^{\frac{\mathrm{i}(1\mp1)\theta}{4}\sigma_3}\begin{pmatrix} 1 & -\mathrm{i}\zeta^{-1} \\ \mathrm{i}\zeta^{-1} & 1 \end{pmatrix} + o(1)\right)\mathrm{e}^{-\mathrm{i}k(\zeta)x\sigma_3},$$

where $k(\zeta) = \frac{1}{2}(\zeta-\zeta^{-1})$*.*

Corollary 2.1.1. *Let $u(x,t)$ be the solution of the Cauchy problem for the* D_fNLSE *and $\Psi(x,t;\zeta)$ the corresponding solution of system* (2) *with the asymptotics stated in Definition* 2.1.1. *Then $\Psi(x,t;\zeta)$ satisfies the symmetry reductions $\sigma_1\overline{\Psi(x,t;\overline{\zeta})}\,\sigma_1=\Psi(x,t;\zeta)$ and $\Psi(x,t;\zeta^{-1})=\zeta\Psi(x,t;\zeta)\sigma_2$.*

Proposition 2.1.2. *Set $\Psi^{\pm}(x,0;\zeta):=\begin{pmatrix}\Psi^{\pm}_{11}(\zeta) & \Psi^{\pm}_{12}(\zeta)\\ \Psi^{\pm}_{21}(\zeta) & \Psi^{\pm}_{22}(\zeta)\end{pmatrix}$. Then $\begin{pmatrix}\Psi^{+}_{12}(\zeta)\\ \Psi^{+}_{22}(\zeta)\end{pmatrix}$ and $\begin{pmatrix}\Psi^{-}_{11}(\zeta)\\ \Psi^{-}_{21}(\zeta)\end{pmatrix}$ have analytic continuation to $\mathbb{C}_+$ (respectively, $\begin{pmatrix}\Psi^{+}_{11}(\zeta)\\ \Psi^{+}_{21}(\zeta)\end{pmatrix}$ and $\begin{pmatrix}\Psi^{-}_{12}(\zeta)\\ \Psi^{-}_{22}(\zeta)\end{pmatrix}$ have analytic continuation to $\mathbb{C}_-$), the monodromy (scattering) matrix, $\mathrm{T}(\zeta)$, is defined by $\Psi^{-}(x,0;\zeta):=\Psi^{+}(x,0;\zeta)\mathrm{T}(\zeta)$, $\mathrm{Im}(\zeta)=0$, where $\mathrm{T}(\zeta)=\begin{pmatrix}a(\zeta) & \overline{b(\overline{\zeta})}\\ b(\zeta) & \overline{a(\overline{\zeta})}\end{pmatrix}$, with $a(\zeta)=(1-\zeta^{-2})^{-1}(\Psi^{+}_{22}(\zeta)\Psi^{-}_{11}(\zeta)-\Psi^{+}_{12}(\zeta)\Psi^{-}_{21}(\zeta))$, $b(\zeta)=(1-\zeta^{-2})^{-1}(\overline{\Psi^{+}_{22}(\overline{\zeta})}\,\Psi^{-}_{21}(\zeta)-\overline{\Psi^{+}_{12}(\overline{\zeta})}\,\Psi^{-}_{11}(\zeta))$, $|a(\zeta)|^2-|b(\zeta)|^2=1$, $a(\zeta^{-1})=\overline{a(\overline{\zeta})}$, $b(\zeta^{-1})=-\overline{b(\overline{\zeta})}$, and $\det(\Psi^{\pm}(x,0;\zeta))|_{\zeta=\pm 1}=0$.*

Corollary 2.1.2. *Let the reflection coefficient associated with the direct scattering problem for the operator $\mathcal{O}^{\mathcal{D}}$ be defined by $r(\zeta):=b(\zeta)/a(\zeta)$. Then $r(\zeta^{-1})=-\overline{r(\overline{\zeta})}$.*

Remark 2.1.2. Note that, even though $a(\zeta)$ (respectively, $\overline{a(\overline{\zeta})}$) has an analytic continuation off $\mathrm{Im}(\zeta)=0$ to $\mathbb{C}_+$ (respectively, $\mathbb{C}_-$) and is continuous on $\overline{\mathbb{C}}_+$ (respectively, $\overline{\mathbb{C}}_-$), $b(\zeta)$ does not, in general, have an analytic continuation to $\mathbb{C}\setminus\mathbb{R}$. Furthermore, for the finite-density initial data considered here, it is shown in [14] that, using Volterra-type integral representations for the elements of $\Psi^{\pm}(x,0;\zeta)$ and a successive approximations argument, $r(\zeta)\in\mathcal{S}_{\mathbb{C}}(\mathbb{R})$ (see, also, Part 1 of [12]).

Lemma 2.1.1. *Let $u(x,t)$ be the solution of the Cauchy problem for the* D_fNLSE *and $\Psi^{\pm}(x,0;\zeta)$ the corresponding (Jost) solutions of $\mathcal{O}^{\mathcal{D}}\Psi^{\pm}(x,0;\zeta)=\mathbf{0}$ given in Definition* 2.1.1. *Then $\Psi^{\pm}(x,0;\zeta)$ have the following asymptotics:*

$$\Psi^{-}(x,0;\zeta)\underset{\zeta\to\infty}{=}\mathrm{e}^{\frac{\mathrm{i}\theta}{2}\sigma_3}\left(\mathrm{I}+\frac{1}{\zeta}\begin{pmatrix}\mathrm{i}\int_{-\infty}^{x}(|u_o(x')|^2-1)\,\mathrm{d}x' & -\mathrm{i}u_o(x)\mathrm{e}^{-\mathrm{i}\theta}\\ \mathrm{i}\overline{u_o(x)}\,\mathrm{e}^{\mathrm{i}\theta} & -\mathrm{i}\int_{-\infty}^{x}(|u_o(x')|^2-1)\,\mathrm{d}x'\end{pmatrix}+\mathcal{O}(\zeta^{-2})\right)\mathrm{e}^{-\mathrm{i}k(\zeta)x\sigma_3},$$

$$\Psi^{+}(x,0;\zeta)\underset{\zeta\to\infty}{=}\left(\mathrm{I}+\frac{1}{\zeta}\begin{pmatrix}\mathrm{i}\int_{+\infty}^{x}(|u_o(x')|^2-1)\,\mathrm{d}x' & -\mathrm{i}u_o(x)\\ \mathrm{i}\overline{u_o(x)} & -\mathrm{i}\int_{+\infty}^{x}(|u_o(x')|^2-1)\,\mathrm{d}x'\end{pmatrix}+\mathcal{O}(\zeta^{-2})\right)\mathrm{e}^{-\mathrm{i}k(\zeta)x\sigma_3},$$

$$\Psi^{-}(x,0;\zeta)\underset{\zeta\to 0}{=}\left(\zeta^{-1}\sigma_2\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}\sigma_3}+\mathcal{O}(1)\right)\mathrm{e}^{-\mathrm{i}k(\zeta)x\sigma_3},\qquad \Psi^{+}(x,0;\zeta)\underset{\zeta\to 0}{=}\left(\zeta^{-1}\sigma_2+\mathcal{O}(1)\right)\mathrm{e}^{-\mathrm{i}k(\zeta)x\sigma_3}.$$

Corollary 2.1.3. *The following asymptotics are valid:*

$$a(\zeta)\underset{\zeta\to\infty}{=}\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}\left(1+\left(\mathrm{i}\int_{-\infty}^{+\infty}(|u_o(x')|^2-1)\,\mathrm{d}x'\right)\zeta^{-1}+\mathcal{O}(\zeta^{-2})\right),\qquad a(\zeta)\underset{\zeta\to 0}{=}\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}}(1+\mathcal{O}(\zeta)),$$
$$r(\zeta)\underset{\zeta\to\infty}{=}\mathcal{O}(\zeta^{-1}),\qquad r(\zeta)\underset{\zeta\to 0}{=}\mathcal{O}(\zeta);$$

in particular, $r(0)=0$.

In [38] it is shown that, for $u(x,0):=u_o(x)$ satisfying $u_o(x)=_{x\to\pm\infty}u_o(\pm\infty)(1+o(1))$, with $u_o(\pm\infty):=\exp(\frac{\mathrm{i}(1\mp 1)\theta}{2})$, $\theta\in[0,2\pi)$ (see Eq. (3)), $u_o(x)\in\mathbf{C}^{\infty}(\mathbb{R})$, and $u_o(x)-u_o(\pm\infty)\in\mathcal{S}_{\mathbb{C}}(\mathbb{R}_{\pm})$, $\sigma_{\mathcal{O}^{\mathcal{D}}}:=\mathrm{spec}(\mathcal{O}^{\mathcal{D}})=\sigma_d\cup\sigma_c$ ($\sigma_d\cap\sigma_c=\emptyset$), where σ_d is the finitely denumerable "discrete" spectrum given by $\sigma_d=\Delta_a\cup\overline{\Delta_a}$, where $\Delta_a:=\{\varsigma_n;\ a(\zeta)|_{\zeta=\varsigma_n}=0,\ \varsigma_n=\mathrm{e}^{\mathrm{i}\phi_n},\ \phi_n\in(0,\pi),\ n\in\{1,2,\dots,N\}\}$, with

$$a(\zeta)=\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}\prod_{n=1}^{N}\frac{(\zeta-\varsigma_n)}{(\zeta-\overline{\varsigma_n})}\exp\left(-\int_{-\infty}^{+\infty}\frac{\ln(1-|r(\mu)|^2)}{(\mu-\zeta)}\frac{\mathrm{d}\mu}{2\pi\mathrm{i}}\right),\quad \zeta\in\mathbb{C}_+,$$
$$0\leqslant\theta=-2\sum_{n=1}^{N}\phi_n-\int_{-\infty}^{+\infty}\frac{\ln(1-|r(\mu)|^2)}{\mu}\frac{\mathrm{d}\mu}{2\pi}<2\pi, \tag{3}$$

and $\Delta_a\cap\overline{\Delta_a}=\emptyset$ ($\mathrm{card}(\sigma_d)=2N$), and σ_c is the "continuous" spectrum given by $\sigma_c=\{\zeta;\ \mathrm{Im}(\zeta)=0\}$, with orientation from $-\infty$ to $+\infty$ ($\mathrm{card}(\sigma_c)=\infty$). Furthermore, it is shown in [38] that, for $r(\zeta)\in\mathcal{S}^1_{\mathbb{C}}(\mathbb{R})$ and $|r(\pm 1)|\neq 1$,

$$a(s+\mathrm{i}\varepsilon)\underset{\varepsilon\downarrow 0}{=}\frac{(-s)^N\exp\left(\mathrm{i}\left(\frac{\theta}{2}+\sum_{n=1}^{N}\phi_n+\mathrm{P.V.}\int_{\mathbb{R}\setminus\{s\}}\frac{\ln(1-|r(\mu)|^2)}{(\mu-s)}\frac{\mathrm{d}\mu}{2\pi}\right)\right)}{(1-|r(s)|^2)^{\kappa_{\mathrm{sgn}(s)}}}(1+o(1)),\quad s\in\{\pm 1\},$$

where P.V. $\int$ denotes the principal value integral, with $\kappa_{\pm}$ real, possibly zero, constants, and (trace identity)

$$\int_{-\infty}^{+\infty}(|u(x',t)|^2-1)\,\mathrm{d}x'=-2\sum_{n=1}^{N}\sin(\phi_n)-\int_{-\infty}^{+\infty}\ln(1-|r(\mu)|^2)\,\tfrac{\mathrm{d}\mu}{2\pi}. \tag{4}$$

The "inverse part" of the ISM analysis is invoked by re-introducing the t-dependence, namely, studying the $\partial_t\Psi(x,t;\zeta)=\mathcal{V}(x,t;\zeta)\Psi(x,t;\zeta)$ component of system (2). The scattering map $(\mathcal{S})$ $u_o(x)\mapsto r(\zeta)=\mathcal{R}(u_o(\cdot))$, which is a bijection for $u_o(x)$ satisfying the finite-density initial conditions and $r(\zeta)\in\mathcal{S}_{\mathbb{C}}^{1}(\mathbb{R})$, linearises the D_fNLSE flow in the sense that, since $a(\zeta,t)=a(\zeta)$ is the "generator" of the integrals of motion and $b(\zeta,t)=b(\zeta)\exp(4\mathrm{i}k(\zeta)\lambda(\zeta)t)$ [12], $r(\zeta,t):=b(\zeta,t)/a(\zeta,t)$ evolves in the scattering data phase space according to the rule $r(\zeta,t)=r(\zeta)\exp(4\mathrm{i}k(\zeta)\lambda(\zeta)t)$. Set [38]

$$\widetilde{\Phi}(x,t;\zeta):=\begin{cases}\begin{pmatrix}\frac{\Psi_{11}^{-}(x,t;\zeta)}{a(\zeta)} & \Psi_{12}^{+}(x,t;\zeta)\\ \frac{\Psi_{21}^{-}(x,t;\zeta)}{a(\zeta)} & \Psi_{22}^{+}(x,t;\zeta)\end{pmatrix}, & \zeta\in\mathbb{C}_+,\\ \begin{pmatrix}\Psi_{11}^{+}(x,t;\zeta) & \frac{\Psi_{12}^{-}(x,t;\zeta)}{\overline{a(\overline{\zeta})}}\\ \Psi_{21}^{+}(x,t;\zeta) & \frac{\Psi_{22}^{-}(x,t;\zeta)}{\overline{a(\overline{\zeta})}}\end{pmatrix}, & \zeta\in\mathbb{C}_-,\end{cases}$$

with $\Psi^{\pm}(x,t;\zeta)$ the solutions of system (2): $\widetilde{\Phi}(x,t;\zeta)$ has the asymptotics [38]

$$\widetilde{\Phi}(x,t;\zeta)\underset{\zeta\to\infty}{=}\left(\mathrm{I}+\frac{1}{\zeta}\begin{pmatrix}\mathrm{i}\int_{+\infty}^{x}(|u(x',t)|^2-1)\,\mathrm{d}x' & -\mathrm{i}u(x,t)\\ \mathrm{i}\overline{u(x,t)} & -\mathrm{i}\int_{+\infty}^{x}(|u(x',t)|^2-1)\,\mathrm{d}x'\end{pmatrix}+\mathcal{O}(\zeta^{-2})\right)\mathrm{e}^{-\mathrm{i}k(\zeta)(x+2\lambda(\zeta)t)\sigma_3},$$
$$\widetilde{\Phi}(x,t;\zeta)\underset{\zeta\to 0}{=}\left(\zeta^{-1}\sigma_2+\mathcal{O}(1)\right)\mathrm{e}^{-\mathrm{i}k(\zeta)(x+2\lambda(\zeta)t)\sigma_3}.$$

Lemma 2.1.2 ([38])**.** *Let $u(x,t)$ be the solution of the Cauchy problem for the* D_fNLSE *with finite-density initial data $u(x,0):=u_o(x)=_{x\to\pm\infty}u_o(\pm\infty)(1+o(1))$, where $u_o(\pm\infty):=\exp(\frac{\mathrm{i}(1\mp1)\theta}{2})$, $0\leqslant\theta=-2\sum_{n=1}^{N}\sin(\phi_n)-\int_{-\infty}^{+\infty}\frac{\ln(1-|r(\mu)|^2)}{\mu}\frac{\mathrm{d}\mu}{2\pi}<2\pi$, $u_o(x)\in\mathbf{C}^{\infty}(\mathbb{R})$, and $u_o(x)-u_o(\pm\infty)\in\mathcal{S}_{\mathbb{C}}(\mathbb{R}_{\pm})$. Set*

$$m(x,t;\zeta):=\widetilde{\Phi}(x,t;\zeta)\exp(\mathrm{i}k(\zeta)(x+2\lambda(\zeta)t)\sigma_3).$$

Then: (1) *the bounded discrete set σ_d is finite;* (2) *the poles of $m(x,t;\zeta)$ are simple;* (3) *the first (respectively, second) column of $m(x,t;\zeta)$ has poles in $\mathbb{C}_+$ (respectively, $\mathbb{C}_-$) at $\{\varsigma_n\}_{n=1}^{N}$ (respectively, $\{\overline{\varsigma_n}\}_{n=1}^{N}$); and* (4) *$m(x,t;\zeta)\colon\mathbb{C}\setminus(\sigma_d\cup\sigma_c)\to\mathrm{M}_2(\mathbb{C})$ solves the following* RHP:

(i) $m(x,t;\zeta)$ is piecewise (sectionally) meromorphic $\forall\,\zeta\in\mathbb{C}\setminus\sigma_c$;

(ii) $m_{\pm}(x,t;\zeta):=\lim_{\substack{\zeta'\to\zeta\\ \pm\mathrm{Im}(\zeta')>0}}m(x,t;\zeta')$ satisfy the jump condition

$$m_+(x,t;\zeta)=m_-(x,t;\zeta)\mathcal{G}(x,t;\zeta),\quad \zeta\in\mathbb{R},$$

where $\mathcal{G}(x,t;\zeta)=\exp(-\mathrm{i}k(\zeta)(x+2\lambda(\zeta)t)\mathrm{ad}(\sigma_3))\begin{pmatrix}1+r(\zeta)r(\zeta^{-1}) & r(\zeta^{-1})\\ r(\zeta) & 1\end{pmatrix}$, and $r(\zeta)$, the reflection coefficient associated with the direct scattering problem for the operator $\mathcal{O}^{\mathcal{D}}$, satisfies $r(\zeta)=_{\zeta\to0}\mathcal{O}(\zeta)$, $r(\zeta)=_{\zeta\to\infty}\mathcal{O}(\zeta^{-1})$, $r(\zeta^{-1})=-\overline{r(\overline{\zeta})}$, and $r(\zeta)\in\mathcal{S}_{\mathbb{C}}^{1}(\mathbb{R})$;

(iii) for the simple poles of $m(x,t;\zeta)$ at $\{\varsigma_n\}_{n=1}^{N}$ and $\{\overline{\varsigma_n}\}_{n=1}^{N}$, there exist nilpotent matrices, with degree of nilpotency 2, such that $m(x,t;\zeta)$ satisfies the polar conditions

$$\mathrm{Res}(m(x,t;\zeta);\varsigma_n)=\lim_{\zeta\to\varsigma_n}m(x,t;\zeta)g_n(x,t)\sigma_-,\qquad n\in\{1,2,\dots,N\},$$
$$\mathrm{Res}(m(x,t;\zeta);\overline{\varsigma_n})=\sigma_1\overline{\mathrm{Res}(m(x,t;\zeta);\varsigma_n)}\,\sigma_1,\qquad n\in\{1,2,\dots,N\},$$

where $g_n(x,t)=g_n\exp(2\mathrm{i}k(\varsigma_n)(x+2\lambda(\varsigma_n)t))$, with

$$g_n:=|\gamma_n|\mathrm{e}^{\mathrm{i}\theta_{\gamma_n}}(\varsigma_n-\overline{\varsigma_n})\exp\left(-\frac{\mathrm{i}\theta}{2}+\int_{-\infty}^{+\infty}\frac{\ln(1-|r(\mu)|^2)}{(\mu-\varsigma_n)}\frac{\mathrm{d}\mu}{2\pi\mathrm{i}}\right)\prod_{\substack{k=1\\k\neq n}}^{N}\left(\frac{\varsigma_n-\overline{\varsigma_k}}{\varsigma_n-\varsigma_k}\right),\qquad \theta_{\gamma_n}=\pm\tfrac{\pi}{2};$$

(iv) $\det(m(x,t;\zeta))|_{\zeta=\pm 1}=0$;

(v) $m(x,t;\zeta)=_{\zeta\to 0}\zeta^{-1}\sigma_2+\mathcal{O}(1)$;

(vi) $m(x,t;\zeta)\underset{\zeta\in\mathbb{C}\setminus(\sigma_d\cup\sigma_c)}{\overset{\zeta\to\infty}{=}}\mathrm{I}+\mathcal{O}(\zeta^{-1})$;

(vii) $m(x,t;\zeta)$ *possesses the symmetry reductions* $m(x,t;\zeta)=\sigma_1\overline{m(x,t;\overline{\zeta})}\,\sigma_1$ *and* $m(x,t;\zeta^{-1})=\zeta m(x,t;\zeta)\sigma_2$.

For $r(\zeta)\in\mathcal{S}^1_{\mathbb{C}}(\mathbb{R})$*: (i) the* RHP *for* $m(x,t;\zeta)$ *formulated above is uniquely asymptotically solvable; and (ii)* $\widetilde{\Phi}(x,t;\zeta)=m(x,t;\zeta)\exp(-\mathrm{i}k(\zeta)(x+2\lambda(\zeta)t)\sigma_3)$ *solves system* (2) *with*

$$u(x,t):=\mathrm{i}\lim_{\substack{\zeta\to\infty\\ \zeta\in\mathbb{C}\setminus(\sigma_d\cup\sigma_c)}}(\zeta(m(x,t;\zeta)-\mathrm{I}))_{12}\tag{5}$$

the solution of the Cauchy problem for the D_fNLSE*, and*

$$\int_{+\infty}^{x}(|u(x',t)|^2-1)\,\mathrm{d}x':=-\mathrm{i}\lim_{\substack{\zeta\to\infty\\ \zeta\in\mathbb{C}\setminus(\sigma_d\cup\sigma_c)}}(\zeta(m(x,t;\zeta)-\mathrm{I}))_{11}.\tag{6}$$

Remark 2.1.3. In this paper, for $r(\zeta)\in\mathcal{S}^1_{\mathbb{C}}(\mathbb{R})$, the solvability of the RHP for $m(x,t;\zeta)$ formulated in Lemma 2.1.2 is proved, via explicit construction, for all sufficiently large $|t|$ ($x/t\sim\mathcal{O}(1)$): the solvability of the RHP in the solitonless sector, $\sigma_d\equiv\emptyset$, for $r(\zeta)\in\mathcal{S}^1_{\mathbb{C}}(\mathbb{R})$, as $|t|\to\infty$ and $|x|\to\infty$ such that $z_o:=x/t\sim\mathcal{O}(1)$ and $\in\mathbb{R}\setminus\{-2,0,2\}$, was proved in [38].

2.2 Summary of Results

In this subsection, the results of this work are summarised in Theorems 2.2.1–2.2.4: before doing so, however, the following preamble is necessary. Recall from Subsection 2.1 that $\varsigma_n:=\mathrm{e}^{\mathrm{i}\phi_n}$, $\phi_n\in(0,\pi)$, $n\in\{1,2,\ldots,N\}$. Set $\varsigma_n:=\xi_n+\mathrm{i}\eta_n$, where $\xi_n=\mathrm{Re}(\varsigma_n)=\cos(\phi_n)\in(-1,1)$, and $\eta_n=\mathrm{Im}(\varsigma_n)=\sin(\phi_n)\in(0,1)$. Throughout this paper, it is assumed that: (1) $\xi_i\neq\xi_j\ \forall\ i\neq j\in\{1,2,\ldots,N\}$; and (2) the following ordering (enumeration) for the elements of the discrete spectrum (solitons), σ_d, is taken, $\xi_1>\xi_2>\cdots>\xi_N$.

Remark 2.2.1. Throughout this paper, the "symbols" $c^{\mathcal{S}}(\diamondsuit)$, $\underline{c}(\flat,\natural,\sharp)$, $\underline{c}(z_1,z_2,z_3,z_4)$, $\underline{c}(\bullet)$, and $\underline{c}$, appearing in the various error estimates, are to be understood as follows: (1) for $\pm\diamondsuit>0$, $c^{\mathcal{S}}(\diamondsuit)\in\mathcal{S}_{\mathbb{C}}(\mathbb{R}_\pm)$; (2) for $\pm\flat>0$, $\underline{c}(\flat,\natural,\sharp)\in\mathcal{L}^\infty_{\mathbb{C}}(\mathbb{R}_\pm\times\mathbb{C}^*\times\overline{\mathbb{C}^*})$, where $\mathbb{C}^*:=\mathbb{C}\setminus\{0\}$; (3) for $(z_1,z_2)\in\mathbb{R}_\pm\times\mathbb{R}_\pm$, $\underline{c}(z_1,z_2,z_3,z_4)\in\mathcal{L}^\infty_{\mathbb{C}}(\mathbb{R}^2_\pm\times\mathbb{C}^*\times\overline{\mathbb{C}^*})$; (4) for $\pm\bullet>0$, $\underline{c}(\bullet)\in\mathcal{L}^\infty_{\mathbb{C}}(\mathrm{D}_\pm)$, where $\mathrm{D}_+:=(0,2)$ and $\mathrm{D}_-:=(-2,0)$; and (5) $\underline{c}\in\mathbb{C}^*$. Even though the symbols $c^{\mathcal{S}}(\diamondsuit)$, $\underline{c}(\flat,\natural,\sharp)$, $\underline{c}(z_1,z_2,z_3,z_4)$, $\underline{c}(\bullet)$, and $\underline{c}$ are not, in general, equal, and should properly be denoted as $c_1(\cdot)$, $c_2(\cdot)$, etc., the simplified notations $c^{\mathcal{S}}(\diamondsuit)$, $\underline{c}(\flat,\natural,\sharp)$, $\underline{c}(z_1,z_2,z_3,z_4)$, $\underline{c}(\bullet)$, and $\underline{c}$ are retained throughout in order to eschew a flood of superfluous notation as well as to maintain consistency with the main theme of this work, namely, to derive explicitly the leading-order asymptotics and the classes to which the errors belong without regard to their precise z_o-dependence.

Remark 2.2.2. In Theorems 2.2.1—2.2.4 below, one should keep, everywhere, the upper (respectively, lower) signs as $t\to+\infty$ (respectively, $t\to-\infty$).

Theorem 2.2.1. *For* $r(\zeta)\in\mathcal{S}^1_{\mathbb{C}}(\mathbb{R})$*, let* $m(x,t;\zeta)$ *be the solution of the Riemann-Hilbert problem formulated in Lemma* 2.1.2*. Let* $u(x,t)$*, the solution of the Cauchy problem for the* D_fNLSE *with finite-density initial data* $u(x,0):=u_o(x)=_{x\to\pm\infty}u_o(\pm\infty)(1+o(1))$*, where* $u_o(\pm\infty):=\exp(\frac{\mathrm{i}(1\mp 1)\theta}{2})$*,* $0\leqslant\theta=-2\sum_{n=1}^N\sin(\phi_n)-\int_{-\infty}^{+\infty}\frac{\ln(1-|r(\mu)|^2)}{\mu}\frac{\mathrm{d}\mu}{2\pi}<2\pi$*,* $u_o(x)\in\mathbf{C}^\infty(\mathbb{R})$*, and* $u_o(x)-u_o(\pm\infty)\in\mathcal{S}_{\mathbb{C}}(\mathbb{R}_\pm)$*, be defined by Eq.* (5)*. Then, for* $\theta_{\gamma_m}=\varepsilon_b\pi/2$*,* $\varepsilon_b\in\{\pm1\}$*,* $m\in\{1,2,\ldots,N\}$*, as* $t\to\pm\infty$ *and* $x\to\mp\infty$ *such that* $z_o:=x/t<-2$ *and* $(x,t)\in\{(x,t);\ x+2t\cos(\phi_m)=\mathcal{O}(1),\ \phi_m\in(0,\pi)\}$*,*

$$\begin{aligned}u(x,t)=\mathrm{e}^{-\mathrm{i}(\theta^\pm(1)+s^\pm)}&\left(\widetilde{u}_{\mathcal{S}}(x,t)+\frac{\sqrt{\nu(\lambda_1)}}{\sqrt{|t|(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}(\widetilde{u}_{\mathcal{C}}(x,t)+\widetilde{u}_{\mathcal{SC}}(x,t))\right.\\&\left.+\mathcal{O}\left(\left(\frac{c^{\mathcal{S}}(\lambda_1)\underline{c}(\lambda_2,\lambda_3,\lambda_4)}{\sqrt{\lambda_1(z_o^2+32)}}+\frac{c^{\mathcal{S}}(\lambda_2)\underline{c}(\lambda_1,\lambda_3,\lambda_4)}{\sqrt{\lambda_2(z_o^2+32)}}\right)\frac{\ln|t|}{(\lambda_1-\lambda_2)t}\right)\right),\end{aligned}\tag{7}$$

where

$$\theta^{+}(j)=\left(\int_{-\infty}^{0}+\int_{\lambda_2}^{\lambda_1}\right)\frac{\ln(1-|r(\mu)|^2)}{\mu^j}\,\frac{\mathrm{d}\mu}{2\pi},\quad j\in\{0,1\},\tag{8}$$

$$\theta^{-}(l)=\left(\int_{0}^{\lambda_2}+\int_{\lambda_1}^{+\infty}\right)\frac{\ln(1-|r(\mu)|^2)}{\mu^l}\,\frac{\mathrm{d}\mu}{2\pi},\quad l\in\{0,1\},\tag{9}$$

$$\begin{gathered}\lambda_1=-\tfrac{1}{2}(a_1-(a_1^2-4)^{1/2}),\qquad \lambda_2=\lambda_1^{-1},\qquad \lambda_3=-\tfrac{1}{2}(a_2-\mathrm{i}(4-a_2^2)^{1/2}),\qquad \lambda_4=\overline{\lambda_3},\\ a_1=\tfrac{1}{4}(z_o-(z_o^2+32)^{1/2}),\qquad a_2=\tfrac{1}{4}(z_o+(z_o^2+32)^{1/2}),\end{gathered}\tag{10}$$

$0<\lambda_2<\lambda_1$, $|\lambda_3|^2=1$, $a_1a_2=-2$,

$$s^{+}=2\sum_{k=m+1}^{N}\phi_k,\qquad s^{-}=2\sum_{k=1}^{m-1}\phi_k,\qquad \nu(z)=-\tfrac{1}{2\pi}\ln(1-|r(z)|^2),\tag{11}$$

$$\widetilde{u}_{\mathcal{S}}(x,t)=\frac{1+\varepsilon_b\widetilde{\varepsilon}_{\mathcal{P}}\mathrm{e}^{-2\mathrm{i}\phi_m+\Omega^{\pm}(x,t)}}{(1+\varepsilon_b\widetilde{\varepsilon}_{\mathcal{P}}\mathrm{e}^{\Omega^{\pm}(x,t)})},\tag{12}$$

$$\widetilde{\varepsilon}_{\mathcal{P}}=\operatorname{sgn}\left(\left(\prod_{k=1}^{m-1}\frac{\sin(\frac{1}{2}(\phi_m+\phi_k))}{\sin(\frac{1}{2}(\phi_m-\phi_k))}\right)\left(\prod_{k=m+1}^{N}\frac{\sin(\frac{1}{2}(\phi_m+\phi_k))}{\sin(\frac{1}{2}(\phi_m-\phi_k))}\right)^{-1}\right)=(-1)^{N-m},\tag{13}$$

$$\Omega^{\pm}(x,t)=-2\sin(\phi_m)(x+2t\cos(\phi_m)-\widetilde{x}_m^{\pm}),\tag{14}$$

$$\begin{aligned}\widetilde{x}_m^{\pm}=&\,\frac{\ln(|\gamma_m|)}{2\sin(\phi_m)}\pm\sum_{k=1}^{N}\frac{\operatorname{sgn}(m-k)}{2\sin(\phi_m)}\ln\left(\left|\frac{\sin(\frac{1}{2}(\phi_m+\phi_k))}{\sin(\frac{1}{2}(\phi_m-\phi_k))}\right|\right)\\ &\pm\frac{1}{2}\left(\int_{0}^{\lambda_2}+\int_{\lambda_1}^{+\infty}-\int_{-\infty}^{0}-\int_{\lambda_2}^{\lambda_1}\right)\frac{\ln(1-|r(\mu)|^2)}{(\mu^2-2\mu\cos(\phi_m)+1)}\,\frac{\mathrm{d}\mu}{2\pi},\end{aligned}\tag{15}$$

$$\widetilde{u}_{\mathcal{C}}(x,t)=\mathrm{i}\mathrm{e}^{\mathrm{i}s^{\pm}}\left(\lambda_1\mathrm{e}^{\mp\mathrm{i}\left(\Theta^{\pm}(z_o,t)\pm(2\mp1)\frac{\pi}{4}\right)}+\lambda_2\mathrm{e}^{\pm\mathrm{i}\left(\Theta^{\pm}(z_o,t)\pm(2\mp1)\frac{\pi}{4}\right)}\right),\tag{16}$$

$$\begin{aligned}\Theta^{\pm}(z_o,t)=&\pm\arg r(\lambda_1)\pm4\sum_{k\in J^{\pm}}\arg(\lambda_1-\mathrm{e}^{\mathrm{i}\phi_k})-\arg\Gamma(\mathrm{i}\nu(\lambda_1))\pm t(\lambda_1-\lambda_2)(z_o+\lambda_1+\lambda_2)\\ &+\nu(\lambda_1)\ln|t|+3\nu(\lambda_1)\ln(\lambda_1-\lambda_2)+\tfrac{1}{2}\nu(\lambda_1)\ln\left(z_o^2+32\right)\mp\Xi^{\pm}(\lambda_1)\pm\tfrac{1}{2}\Xi^{\pm}(0),\end{aligned}\tag{17}$$

$$\Xi^{+}(z)=\frac{1}{\pi}\left(\int_{-\infty}^{0}+\int_{\lambda_2}^{\lambda_1}\right)\ln|\mu-z|\,\mathrm{d}\ln(1-|r(\mu)|^2),\tag{18}$$

$$\Xi^{-}(z)=\frac{1}{\pi}\left(\int_{0}^{\lambda_2}+\int_{\lambda_1}^{+\infty}\right)\ln|\mu-z|\,\mathrm{d}\ln(1-|r(\mu)|^2),\tag{19}$$

$\sum_{k\in J^{+}}:=\sum_{k=m+1}^{N}$, $\sum_{k\in J^{-}}:=\sum_{k=1}^{m-1}$, $\Gamma(\cdot)$ *is the gamma function* [51], *and*

$$\widetilde{u}_{\mathcal{SC}}(x,t)=\sum_{k=1}^{7}\widetilde{u}_{\mathcal{SC}}^{(k)}(x,t),\tag{20}$$

with

$$\begin{aligned}\widetilde{u}_{\mathcal{SC}}^{(1)}(x,t)=&\,-2\mathrm{i}\varepsilon_b\widetilde{\varepsilon}_{\mathcal{P}}\csc(\phi_m)\sin(s^{\pm})\cos(\Theta^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})\sinh(\Omega^{\pm}(x,t)),\\ \widetilde{u}_{\mathcal{SC}}^{(2)}(x,t)=&\,2\mathrm{i}\varepsilon_b\widetilde{\varepsilon}_{\mathcal{P}}\left(\cos(\phi_m)\mathrm{e}^{\mathrm{i}s^{\pm}}+2\sin(\phi_m)\sin(s^{\pm})\right)\cos(\Theta^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})\mathrm{e}^{\Omega^{\pm}(x,t)},\\ \widetilde{u}_{\mathcal{SC}}^{(3)}(x,t)=&\,\frac{4\mathrm{i}\varepsilon_b\widetilde{\varepsilon}_{\mathcal{P}}\lambda_1^2\sin(\phi_m)\sin(s^{\pm})\mathrm{e}^{\Omega^{\pm}(x,t)}}{(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)^2}\left(((\lambda_1+\lambda_2)\cos(\phi_m)-2)\cos(\Theta^{\pm}(z_o,t)\right.\\ &\left.\pm(2\mp1)\tfrac{\pi}{4})\pm(\lambda_1-\lambda_2)\sin(\phi_m)\sin(\Theta^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})\right),\end{aligned}$$

$$\begin{aligned}
\widetilde{u}^{(4)}_{\mathcal{SC}}(x,t) &= \frac{2\mathrm{i}\varepsilon_b\widetilde{\varepsilon}_{\mathcal{P}}\lambda_1\cos(\phi_m)\mathrm{e}^{\Omega^{\pm}(x,t)}}{(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)}\big(2\cos(s^{\pm}-\phi_m)\cos(\Theta^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})-(\lambda_1+\lambda_2)\\
&\quad\times\cos(s^{\pm})\cos(\Theta^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})\mp(\lambda_1-\lambda_2)\sin(s^{\pm})\sin(\Theta^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})\big),\\
\widetilde{u}^{(5)}_{\mathcal{SC}}(x,t) &= -\frac{4\varepsilon_b\widetilde{\varepsilon}_{\mathcal{P}}\sin(\phi_m)\mathrm{e}^{\Omega^{\pm}(x,t)}}{(1-\mathrm{e}^{2\Omega^{\pm}(x,t)})}\cos(\Theta^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})\Big(\mathrm{e}^{-\mathrm{i}s^{\pm}}+\cos(s^{\pm}-\phi_m)\mathrm{e}^{-\mathrm{i}\phi_m+2\Omega^{\pm}(x,t)}\Big),\\
\widetilde{u}^{(6)}_{\mathcal{SC}}(x,t) &= \frac{4\widetilde{\varepsilon}_{\mathcal{P}}\lambda_1\sin(\phi_m)\mathrm{e}^{\Omega^{\pm}(x,t)}}{(1-\mathrm{e}^{2\Omega^{\pm}(x,t)})(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)}\Big(\big(\mathrm{e}^{\Omega^{\pm}(x,t)}(1+\varepsilon_b\widetilde{\varepsilon}_{\mathcal{P}}\cos(\phi_m)\mathrm{e}^{-\mathrm{i}\phi_m+\Omega^{\pm}(x,t)})\big)\\
&\quad\times\big(-2\widetilde{\varepsilon}_{\mathcal{P}}\cos(s^{\pm}-\phi_m)\cos(\Theta^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})+\widetilde{\varepsilon}_{\mathcal{P}}(\lambda_1+\lambda_2)\cos(s^{\pm})\cos(\Theta^{\pm}(z_o,t)\\
&\quad\pm(2\mp1)\tfrac{\pi}{4})\pm\widetilde{\varepsilon}_{\mathcal{P}}(\lambda_1-\lambda_2)\sin(s^{\pm})\sin(\Theta^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})\big)+\Big(1-\varepsilon_b\widetilde{\varepsilon}_{\mathcal{P}}\mathrm{e}^{\Omega^{\pm}(x,t)}\Big)(2\mathrm{i}\varepsilon_b\\
&\quad\times\sin(s^{\pm}-\phi_m)\cos(\Theta^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})-\mathrm{i}\varepsilon_b(\lambda_1+\lambda_2)\sin(s^{\pm})\cos(\Theta^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})\\
&\quad\pm\mathrm{i}\varepsilon_b(\lambda_1-\lambda_2)\cos(s^{\pm})\sin(\Theta^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4}))\Big),\\
\widetilde{u}^{(7)}_{\mathcal{SC}}(x,t) &= -\frac{8\lambda_1^2\sin^2(\phi_m)\mathrm{e}^{-\mathrm{i}\phi_m+2\Omega^{\pm}(x,t)}}{(1-\mathrm{e}^{2\Omega^{\pm}(x,t)})(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)^2}\big(((\lambda_1+\lambda_2)\cos(\phi_m)-2)\cos(\Theta^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})\\
&\quad\pm(\lambda_1-\lambda_2)\sin(\phi_m)\sin(\Theta^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})\big)\Big((1+\varepsilon_b\widetilde{\varepsilon}_{\mathcal{P}}\mathrm{e}^{\Omega^{\pm}(x,t)})\sin(s^{\pm})+\mathrm{i}\left(\tfrac{1-\varepsilon_b\widetilde{\varepsilon}_{\mathcal{P}}\mathrm{e}^{\Omega^{\pm}(x,t)}}{1+\varepsilon_b\widetilde{\varepsilon}_{\mathcal{P}}\mathrm{e}^{\Omega^{\pm}(x,t)}}\right)\\
&\quad\times\cos(s^{\pm})\Big).
\end{aligned}$$

For the conditions stated in the formulation of the Theorem, as $t\to\pm\infty$ *and* $x\to\pm\infty$ *such that* $z_o>2$ *and* $(x,t)\in\{(x,t);\ x+2t\cos(\phi_m)=\mathcal{O}(1),\ \phi_m\in(-\pi,0)\}$,

$$\begin{aligned}
u(x,t)=-\mathrm{e}^{-\mathrm{i}(\psi^{\pm}(1)+s^{\pm})}\Bigg(&\widehat{u}_{\mathcal{S}}(x,t)+\frac{\sqrt{\nu(\aleph_4)}}{\sqrt{|t|(\aleph_3-\aleph_4)}\,(z_o^2+32)^{1/4}}(\widehat{u}_{\mathcal{C}}(x,t)+\widehat{u}_{\mathcal{SC}}(x,t))\\
&+\mathcal{O}\left(\left(\frac{c^{\mathcal{S}}(\aleph_3)\underline{c}(\aleph_4,\aleph_1,\aleph_2)}{\sqrt{|\aleph_3|(z_o^2+32)}}+\frac{c^{\mathcal{S}}(\aleph_4)\underline{c}(\aleph_3,\aleph_1,\aleph_2)}{\sqrt{|\aleph_4|(z_o^2+32)}}\right)\frac{\ln|t|}{(\aleph_3-\aleph_4)t}\right)\Bigg),
\end{aligned}\tag{21}$$

where

$$\psi^{+}(j)=\left(\int_{-\infty}^{\aleph_4}+\int_{\aleph_3}^{0}\right)\frac{\ln(1-|r(\mu)|^2)}{\mu^j}\,\frac{\mathrm{d}\mu}{2\pi},\quad j\in\{0,1\},\tag{22}$$

$$\psi^{-}(l)=\left(\int_{\aleph_4}^{\aleph_3}+\int_{0}^{+\infty}\right)\frac{\ln(1-|r(\mu)|^2)}{\mu^l}\,\frac{\mathrm{d}\mu}{2\pi},\quad l\in\{0,1\},\tag{23}$$

$$\aleph_1=-\tfrac{1}{2}(a_1-\mathrm{i}(4-a_1^2)^{1/2}),\qquad \aleph_2=\overline{\aleph_1},\qquad \aleph_3=-\tfrac{1}{2}(a_2-(a_2^2-4)^{1/2}),\qquad \aleph_4=\aleph_3^{-1},\tag{24}$$

$\aleph_4<\aleph_3<0$, $|\aleph_1|^2=1$,

$$\widehat{u}_{\mathcal{S}}(x,t)=\frac{1+\varepsilon_b\widehat{\varepsilon}_{\mathcal{P}}\mathrm{e}^{-2\mathrm{i}\phi_m+\mho^{\pm}(x,t)}}{(1+\varepsilon_b\widehat{\varepsilon}_{\mathcal{P}}\mathrm{e}^{\mho^{\pm}(x,t)})},\tag{25}$$

$$\widehat{\varepsilon}_{\mathcal{P}}=\operatorname{sgn}\left(\left(\prod_{k=1}^{m-1}\frac{(-\sin(\frac{1}{2}(\phi_m+\phi_k)))}{\sin(\frac{1}{2}(\phi_m-\phi_k))}\right)\left(\prod_{k=m+1}^{N}\frac{(-\sin(\frac{1}{2}(\phi_m+\phi_k)))}{\sin(\frac{1}{2}(\phi_m-\phi_k))}\right)^{-1}\right)=(-1)^{m-1},\tag{26}$$

$$\mho^{\pm}(x,t)=-2\sin(\phi_m)(x+2t\cos(\phi_m)-\widehat{x}_m^{\pm}),\tag{27}$$

$$\begin{aligned}
\widehat{x}_m^{\pm}=&\frac{\ln(|\gamma_m|)}{2\sin(\phi_m)}\pm\sum_{k=1}^{N}\frac{\operatorname{sgn}(m-k)}{2\sin(\phi_m)}\ln\left(\left|\frac{\sin(\frac{1}{2}(\phi_m+\phi_k))}{\sin(\frac{1}{2}(\phi_m-\phi_k))}\right|\right)\\
&\pm\frac{1}{2}\left(\int_{-\infty}^{\aleph_4}+\int_{\aleph_3}^{0}-\int_{\aleph_4}^{\aleph_3}-\int_{0}^{+\infty}\right)\frac{\ln(1-|r(\mu)|^2)}{(\mu^2+2\mu\cos(\phi_m)+1)}\,\frac{\mathrm{d}\mu}{2\pi},
\end{aligned}\tag{28}$$

$$\widehat{u}_{\mathcal{C}}(x,t)=\mathrm{i}\mathrm{e}^{\mathrm{i}s^{\pm}}\left(\aleph_3\mathrm{e}^{\pm\mathrm{i}\left(\Phi^{\pm}(z_o,t)\pm(2\mp1)\frac{\pi}{4}\right)}+\aleph_4\mathrm{e}^{\mp\mathrm{i}\left(\Phi^{\pm}(z_o,t)\pm(2\mp1)\frac{\pi}{4}\right)}\right),\tag{29}$$

$$\Phi^{\pm}(z_o,t)=\pm\arg r(\aleph_4)\pm 4\sum_{k\in J^{\pm}}\arg(\aleph_4+\mathrm{e}^{\mathrm{i}\phi_k})-\arg\Gamma(\mathrm{i}\nu(\aleph_4))\pm t(\aleph_4-\aleph_3)(z_o+\aleph_3+\aleph_4)$$
$$+\nu(\aleph_4)\ln|t|+3\nu(\aleph_4)\ln(\aleph_3-\aleph_4)+\tfrac{1}{2}\nu(\aleph_4)\ln\big(z_o^2+32\big)\mp\Lambda^{\pm}(\aleph_4)\pm\tfrac{1}{2}\Lambda^{\pm}(0), \tag{30}$$

$$\Lambda^{+}(z)=\frac{1}{\pi}\left(\int_{-\infty}^{\aleph_4}+\int_{\aleph_3}^{0}\right)\ln|\mu-z|\,\mathrm{d}\ln(1-|r(\mu)|^2), \tag{31}$$

$$\Lambda^{-}(z)=\frac{1}{\pi}\left(\int_{\aleph_4}^{\aleph_3}+\int_{0}^{+\infty}\right)\ln|\mu-z|\,\mathrm{d}\ln(1-|r(\mu)|^2), \tag{32}$$

and

$$\widehat{u}_{\mathcal{SC}}(x,t)=\sum_{k=1}^{7}\widehat{u}_{\mathcal{SC}}^{(k)}(x,t), \tag{33}$$

with

$$\widehat{u}_{\mathcal{SC}}^{(1)}(x,t)=2\mathrm{i}\varepsilon_b\widehat{\varepsilon}_{\mathcal{P}}\csc(\phi_m)\sin(s^{\pm})\cos(\Phi^{\pm}(z_o,t)\pm(2\mp 1)\tfrac{\pi}{4})\sinh(\mho^{\pm}(x,t)),$$
$$\widehat{u}_{\mathcal{SC}}^{(2)}(x,t)=-2\mathrm{i}\varepsilon_b\widehat{\varepsilon}_{\mathcal{P}}\Big(\cos(\phi_m)\mathrm{e}^{\mathrm{i}s^{\pm}}+2\sin(\phi_m)\sin(s^{\pm})\Big)\cos(\Phi^{\pm}(z_o,t)\pm(2\mp 1)\tfrac{\pi}{4})\mathrm{e}^{\mho^{\pm}(x,t)},$$
$$\widehat{u}_{\mathcal{SC}}^{(3)}(x,t)=\frac{4\mathrm{i}\varepsilon_b\widehat{\varepsilon}_{\mathcal{P}}\aleph_4^2\sin(\phi_m)\sin(s^{\pm})\mathrm{e}^{\mho^{\pm}(x,t)}}{(\aleph_4^2+2\aleph_4\cos(\phi_m)+1)^2}\big(((\aleph_4+\aleph_3)\cos(\phi_m)+2)\cos(\Phi^{\pm}(z_o,t)$$
$$\pm(2\mp 1)\tfrac{\pi}{4})\pm(\aleph_4-\aleph_3)\sin(\phi_m)\sin(\Phi^{\pm}(z_o,t)\pm(2\mp 1)\tfrac{\pi}{4})\big),$$
$$\widehat{u}_{\mathcal{SC}}^{(4)}(x,t)=\frac{2\mathrm{i}\varepsilon_b\widehat{\varepsilon}_{\mathcal{P}}\aleph_4\cos(\phi_m)\mathrm{e}^{\mho^{\pm}(x,t)}}{(\aleph_4^2+2\aleph_4\cos(\phi_m)+1)}\big(2\cos(s^{\pm}-\phi_m)\cos(\Phi^{\pm}(z_o,t)\pm(2\mp 1)\tfrac{\pi}{4})+(\aleph_4+\aleph_3)\cos(s^{\pm})$$
$$\times\cos(\Phi^{\pm}(z_o,t)\pm(2\mp 1)\tfrac{\pi}{4})\pm(\aleph_4-\aleph_3)\sin(s^{\pm})\sin(\Phi^{\pm}(z_o,t)\pm(2\mp 1)\tfrac{\pi}{4})\big),$$
$$\widehat{u}_{\mathcal{SC}}^{(5)}(x,t)=\frac{4\varepsilon_b\widehat{\varepsilon}_{\mathcal{P}}\sin(\phi_m)\mathrm{e}^{\mho^{\pm}(x,t)}}{(1-\mathrm{e}^{2\mho^{\pm}(x,t)})}\cos(\Phi^{\pm}(z_o,t)\pm(2\mp 1)\tfrac{\pi}{4})\Big(\mathrm{e}^{-\mathrm{i}s^{\pm}}+\cos(s^{\pm}-\phi_m)\mathrm{e}^{-\mathrm{i}\phi_m+2\mho^{\pm}(x,t)}\Big),$$
$$\widehat{u}_{\mathcal{SC}}^{(6)}(x,t)=-\frac{4\widehat{\varepsilon}_{\mathcal{P}}\aleph_4\sin(\phi_m)\mathrm{e}^{\mho^{\pm}(x,t)}}{(1-\mathrm{e}^{2\mho^{\pm}(x,t)})(\aleph_4^2+2\aleph_4\cos(\phi_m)+1)}\Big(\Big(\mathrm{e}^{\mho^{\pm}(x,t)}(1+\varepsilon_b\widehat{\varepsilon}_{\mathcal{P}}\cos(\phi_m)\mathrm{e}^{-\mathrm{i}\phi_m+\mho^{\pm}(x,t)})\Big)$$
$$\times\big(2\widehat{\varepsilon}_{\mathcal{P}}\cos(s^{\pm}-\phi_m)\cos(\Phi^{\pm}(z_o,t)\pm(2\mp 1)\tfrac{\pi}{4})+\widehat{\varepsilon}_{\mathcal{P}}(\aleph_4+\aleph_3)\cos(s^{\pm})\cos(\Phi^{\pm}(z_o,t)$$
$$\pm(2\mp 1)\tfrac{\pi}{4})\pm\widehat{\varepsilon}_{\mathcal{P}}(\aleph_4-\aleph_3)\sin(s^{\pm})\sin(\Phi^{\pm}(z_o,t)\pm(2\mp 1)\tfrac{\pi}{4})\big)-\Big(1-\varepsilon_b\widehat{\varepsilon}_{\mathcal{P}}\mathrm{e}^{\mho^{\pm}(x,t)}\Big)\big(2\mathrm{i}\varepsilon_b$$
$$\times\sin(s^{\pm}-\phi_m)\cos(\Phi^{\pm}(z_o,t)\pm(2\mp 1)\tfrac{\pi}{4})+\mathrm{i}\varepsilon_b(\aleph_4+\aleph_3)\sin(s^{\pm})\cos(\Phi^{\pm}(z_o,t)\pm(2\mp 1)\tfrac{\pi}{4})$$
$$\mp\mathrm{i}\varepsilon_b(\aleph_4-\aleph_3)\cos(s^{\pm})\sin(\Phi^{\pm}(z_o,t)\pm(2\mp 1)\tfrac{\pi}{4})\big)\Big),$$
$$\widehat{u}_{\mathcal{SC}}^{(7)}(x,t)=-\frac{8\aleph_4^2\sin^2(\phi_m)\mathrm{e}^{-\mathrm{i}\phi_m+2\mho^{\pm}(x,t)}}{(1-\mathrm{e}^{2\mho^{\pm}(x,t)})(\aleph_4^2+2\aleph_4\cos(\phi_m)+1)^2}\big(((\aleph_4+\aleph_3)\cos(\phi_m)+2)\cos(\Phi^{\pm}(z_o,t)\pm(2\mp 1)\tfrac{\pi}{4})$$
$$\pm(\aleph_4-\aleph_3)\sin(\phi_m)\sin(\Phi^{\pm}(z_o,t)\pm(2\mp 1)\tfrac{\pi}{4})\big)\Big((1+\varepsilon_b\widehat{\varepsilon}_{\mathcal{P}}\mathrm{e}^{\mho^{\pm}(x,t)})\sin(s^{\pm})+\mathrm{i}\left(\tfrac{1-\varepsilon_b\widehat{\varepsilon}_{\mathcal{P}}\mathrm{e}^{\mho^{\pm}(x,t)}}{1+\varepsilon_b\widehat{\varepsilon}_{\mathcal{P}}\mathrm{e}^{\mho^{\pm}(x,t)}}\right)$$
$$\times\cos(s^{\pm})\Big).$$

Theorem 2.2.2. *For $r(\zeta)\in\mathcal{S}_{\mathbb{C}}^{1}(\mathbb{R})$, let $m(x,t;\zeta)$ be the solution of the Riemann-Hilbert problem formulated in Lemma 2.1.2. Let $u(x,t)$, the solution of the Cauchy problem for the* D_fNLSE *with finite-density initial data $u(x,0):=u_o(x)=_{x\to\pm\infty}u_o(\pm\infty)(1+o(1))$, where $u_o(\pm\infty):=\exp(\frac{\mathrm{i}(1\mp 1)\theta}{2})$, $0\leqslant\theta=-2\sum_{n=1}^{N}\sin(\phi_n)-\int_{-\infty}^{+\infty}\frac{\ln(1-|r(\mu)|^2)}{\mu}\frac{\mathrm{d}\mu}{2\pi}<2\pi$, $u_o(x)\in\mathbf{C}^{\infty}(\mathbb{R})$, and $u_o(x)-u_o(\pm\infty)\in\mathcal{S}_{\mathbb{C}}(\mathbb{R}_{\pm})$, be defined by Eq. (5), and $\int_{+\infty}^{x}(|u(x',t)|^2-1)\,\mathrm{d}x'$ be defined by Eq. (6). Let $\epsilon\in\{\pm 1\}$. Then, for $\theta_{\gamma_m}=\varepsilon_b\pi/2$, $\varepsilon_b\in\{\pm 1\}$, $m\in\{1,2,\dots,N\}$, as $t\to\pm\infty$ and $x\to\mp\infty$ such that $z_o<-2$ and $(x,t)\in\{(x,t);\,x+2t\cos(\phi_m)=\mathcal{O}(1),\ \phi_m\in(0,\pi)\}$,*

$$\int_{\mathrm{sgn}(\epsilon)\infty}^{x}(|u(x',t)|^2-1)\,\mathrm{d}x'=\widetilde{\mathbb{S}}_{\epsilon}^{\pm}+\widetilde{\mathcal{H}}_{\epsilon}^{\pm}+\widetilde{\mathcal{E}}_{\mathcal{S}}(x,t)+\frac{\sqrt{\nu(\lambda_1)}}{\sqrt{|t|(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\Big(\widetilde{\mathcal{E}}_{\mathcal{C}}(x,t)+\widetilde{\mathcal{E}}_{\mathcal{SC}}(x,t)\Big)$$
$$+\mathcal{O}\left(\left(\frac{c^{\mathcal{S}}(\lambda_1)\underline{c}(\lambda_2,\lambda_3,\lambda_4)}{\sqrt{\lambda_1(z_o^2+32)}}+\frac{c^{\mathcal{S}}(\lambda_2)\underline{c}(\lambda_1,\lambda_3,\lambda_4)}{\sqrt{\lambda_2(z_o^2+32)}}\right)\frac{\ln|t|}{(\lambda_1-\lambda_2)t}\right), \tag{34}$$

where

$$\widetilde{\mathbb{S}}_{\epsilon}^{+}=\begin{cases}2\sum_{k=m+1}^{N}\sin(\phi_k), & \epsilon=+1,\\ -2\sum_{k=1}^{m}\sin(\phi_k), & \epsilon=-1,\end{cases}\qquad \widetilde{\mathbb{S}}_{\epsilon}^{-}=\begin{cases}2\sum_{k=1}^{m-1}\sin(\phi_k), & \epsilon=+1,\\ -2\sum_{k=m}^{N}\sin(\phi_k), & \epsilon=-1,\end{cases} \tag{35}$$

$$\widetilde{\mathcal{H}}_{\epsilon}^{+}=\begin{cases}\theta^{+}(0), & \epsilon=+1,\\ -\theta^{-}(0), & \epsilon=-1,\end{cases}\qquad \widetilde{\mathcal{H}}_{\epsilon}^{-}=\begin{cases}\theta^{-}(0), & \epsilon=+1,\\ -\theta^{+}(0), & \epsilon=-1,\end{cases} \tag{36}$$

$$\widetilde{\mathcal{E}}_{\mathcal{S}}(x,t)=\frac{2\varepsilon_b\widetilde{\varepsilon}_{\mathcal{P}}\sin(\phi_m)\mathrm{e}^{\Omega^{\pm}(x,t)}}{(1+\varepsilon_b\widetilde{\varepsilon}_{\mathcal{P}}\mathrm{e}^{\Omega^{\pm}(x,t)})}, \tag{37}$$

$$\widetilde{\mathcal{E}}_{\mathcal{C}}(x,t)=-2\cos(s^{\pm})\cos\big(\Theta^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4}\big)\,, \tag{38}$$

and

$$\widetilde{\mathcal{E}}_{\mathcal{SC}}(x,t)=\sum_{k=1}^{7}\widetilde{\mathcal{E}}_{\mathcal{SC}}^{(k)}(x,t), \tag{39}$$

with

$$\begin{aligned}
\widetilde{\mathcal{E}}_{\mathcal{SC}}^{(1)}(x,t) &= \frac{8\lambda_1^2\sin^2(\phi_m)\cos(s^{\pm})\mathrm{e}^{2\Omega^{\pm}(x,t)}}{(1+\varepsilon_b\widetilde{\varepsilon}_{\mathcal{P}}\mathrm{e}^{\Omega^{\pm}(x,t)})^2(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)^2}\big(((\lambda_1+\lambda_2)\cos(\phi_m)-2)\cos(\Theta^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})\\
&\quad\pm(\lambda_1-\lambda_2)\sin(\phi_m)\sin(\Theta^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})\big)\,,\\
\widetilde{\mathcal{E}}_{\mathcal{SC}}^{(2)}(x,t) &= \frac{4\lambda_1\widetilde{\varepsilon}_{\mathcal{P}}\sin(\phi_m)\mathrm{e}^{\Omega^{\pm}(x,t)}}{(1-\mathrm{e}^{2\Omega^{\pm}(x,t)})(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)}\Big(2(\widetilde{\varepsilon}_{\mathcal{P}}\cos(\phi_m)\sin(s^{\pm}-\phi_m)\mathrm{e}^{\Omega^{\pm}(x,t)}-\varepsilon_b\sin(s^{\pm}))\\
&\quad\times\cos(\Theta^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})-(\lambda_1+\lambda_2)(\widetilde{\varepsilon}_{\mathcal{P}}\cos(\phi_m)\sin(s^{\pm})\mathrm{e}^{\Omega^{\pm}(x,t)}-\varepsilon_b\sin(s^{\pm}+\phi_m))\\
&\quad\times\cos(\Theta^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})\pm(\lambda_1-\lambda_2)(\widetilde{\varepsilon}_{\mathcal{P}}\cos(\phi_m)\cos(s^{\pm})\mathrm{e}^{\Omega^{\pm}(x,t)}-\varepsilon_b\cos(s^{\pm}+\phi_m))\\
&\quad\times\sin(\Theta^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})\big)\,,\\
\widetilde{\mathcal{E}}_{\mathcal{SC}}^{(3)}(x,t) &= \frac{2\varepsilon_b\widetilde{\varepsilon}_{\mathcal{P}}\lambda_1\cos(\phi_m)\mathrm{e}^{\Omega^{\pm}(x,t)}}{(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)}\big(2\cos(s^{\pm}-\phi_m)\cos(\Theta^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})-(\lambda_1+\lambda_2)\cos(s^{\pm})\\
&\quad\times\cos(\Theta^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})\mp(\lambda_1-\lambda_2)\sin(s^{\pm})\sin(\Theta^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})\big)\,,\\
\widetilde{\mathcal{E}}_{\mathcal{SC}}^{(4)}(x,t) &= \frac{4\varepsilon_b\widetilde{\varepsilon}_{\mathcal{P}}\lambda_1^2\sin(\phi_m)\sin(s^{\pm})\mathrm{e}^{\Omega^{\pm}(x,t)}}{(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)^2}\big(((\lambda_1+\lambda_2)\cos(\phi_m)-2)\cos(\Theta^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})\\
&\quad\pm(\lambda_1-\lambda_2)\sin(\phi_m)\sin(\Theta^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})\big)\,,\\
\widetilde{\mathcal{E}}_{\mathcal{SC}}^{(5)}(x,t) &= -2\varepsilon_b\widetilde{\varepsilon}_{\mathcal{P}}\csc(\phi_m)\sin(s^{\pm})\cos(\Theta^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})\sinh(\Omega^{\pm}(x,t)),\\
\widetilde{\mathcal{E}}_{\mathcal{SC}}^{(6)}(x,t) &= \frac{4\sin(\phi_m)\sin(s^{\pm}-\phi_m)}{(1-\mathrm{e}^{2\Omega^{\pm}(x,t)})}\cos(\Theta^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})\mathrm{e}^{2\Omega^{\pm}(x,t)},\\
\widetilde{\mathcal{E}}_{\mathcal{SC}}^{(7)}(x,t) &= 2\varepsilon_b\widetilde{\varepsilon}_{\mathcal{P}}\cos(s^{\pm}-\phi_m)\cos(\Theta^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})\mathrm{e}^{\Omega^{\pm}(x,t)},
\end{aligned}$$

and $\theta^{\pm}(\cdot)$, $\{\lambda_n\}_{n=1}^{4}$, $s^{\pm}$ *and* $\nu(\cdot)$, $\widetilde{\varepsilon}_{\mathcal{P}}$, $\Omega^{\pm}(x,t)$, *and* $\Theta^{\pm}(z_o,t)$ *given in Theorem* 2.2.1, *Eqs.* (8)–(9), (10), (11), (13), (14)–(15), *and* (17)–(19), *respectively.*

For the conditions stated in the formulation of the Theorem, as $t\to\pm\infty$ *and* $x\to\pm\infty$ *such that* $z_o>2$ *and* $(x,t)\in\{(x,t);\ x+2t\cos(\phi_m)=\mathcal{O}(1),\ \phi_m\in(-\pi,0)\}$,

$$\begin{aligned}
\int_{\operatorname{sgn}(\epsilon)\infty}^{x}(|u(x',t)|^2-1)\,\mathrm{d}x' &= \widehat{\mathbb{S}}_{\epsilon}^{\pm}+\widehat{\mathcal{H}}_{\epsilon}^{\pm}+\widehat{\mathcal{E}}_{\mathcal{S}}(x,t)+\frac{\sqrt{\nu(\aleph_4)}}{\sqrt{|t|(\aleph_3-\aleph_4)}\,(z_o^2+32)^{1/4}}\Big(\widehat{\mathcal{E}}_{\mathcal{C}}(x,t)+\widehat{\mathcal{E}}_{\mathcal{SC}}(x,t)\Big)\\
&\quad+\mathcal{O}\left(\left(\frac{c^{\mathcal{S}}(\aleph_3)\underline{c}(\aleph_4,\aleph_1,\aleph_2)}{\sqrt{|\aleph_3|(z_o^2+32)}}+\frac{c^{\mathcal{S}}(\aleph_4)\underline{c}(\aleph_3,\aleph_1,\aleph_2)}{\sqrt{|\aleph_4|(z_o^2+32)}}\right)\frac{\ln|t|}{(\aleph_3-\aleph_4)t}\right),
\end{aligned} \tag{40}$$

where

$$\widehat{\mathbb{S}}_{\epsilon}^{+}=\begin{cases}-2\sum_{k=1}^{m}\sin(\phi_k), & \epsilon=+1,\\ 2\sum_{k=m+1}^{N}\sin(\phi_k), & \epsilon=-1,\end{cases} \qquad \widehat{\mathbb{S}}_{\epsilon}^{-}=\begin{cases}-2\sum_{k=m}^{N}\sin(\phi_k), & \epsilon=+1,\\ 2\sum_{k=1}^{m-1}\sin(\phi_k), & \epsilon=-1,\end{cases} \tag{41}$$

$$\widehat{\mathcal{H}}_{\epsilon}^{+}=\begin{cases}\psi^{-}(0), & \epsilon=+1,\\ -\psi^{+}(0), & \epsilon=-1,\end{cases} \qquad \widehat{\mathcal{H}}_{\epsilon}^{-}=\begin{cases}\psi^{+}(0), & \epsilon=+1,\\ -\psi^{-}(0), & \epsilon=-1,\end{cases} \tag{42}$$

$$\widehat{\mathcal{E}}_{\mathcal{S}}(x,t)=\frac{2\varepsilon_b\widehat{\varepsilon}_{\mathcal{P}}\sin(\phi_m)\mathrm{e}^{\mho^{\pm}(x,t)}}{(1+\varepsilon_b\widehat{\varepsilon}_{\mathcal{P}}\mathrm{e}^{\mho^{\pm}(x,t)})}, \tag{43}$$

$$\widehat{\mathcal{E}}_{\mathcal{C}}(x,t)=2\cos(s^{\pm})\cos\big(\Phi^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4}\big), \tag{44}$$

and

$$\widehat{\mathcal{E}}_{\mathcal{SC}}(x,t)=\sum_{k=1}^{7}\widehat{\mathcal{E}}_{\mathcal{SC}}^{(k)}(x,t), \tag{45}$$

with

$$\begin{aligned}
\widehat{\mathcal{E}}_{\mathcal{SC}}^{(1)}(x,t) &= \frac{8\aleph_4^2\sin^2(\phi_m)\cos(s^{\pm})\mathrm{e}^{2\mho^{\pm}(x,t)}}{(1+\varepsilon_b\widehat{\varepsilon}_{\mathcal{P}}\mathrm{e}^{\mho^{\pm}(x,t)})^2(\aleph_4^2+2\aleph_4\cos(\phi_m)+1)^2}\big(((\aleph_4+\aleph_3)\cos(\phi_m)+2)\cos(\Phi^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})\\
&\quad\pm(\aleph_4-\aleph_3)\sin(\phi_m)\sin(\Phi^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})\big),\\
\widehat{\mathcal{E}}_{\mathcal{SC}}^{(2)}(x,t) &= \frac{4\aleph_4\widehat{\varepsilon}_{\mathcal{P}}\sin(\phi_m)\mathrm{e}^{\mho^{\pm}(x,t)}}{(1-\mathrm{e}^{2\mho^{\pm}(x,t)})(\aleph_4^2+2\aleph_4\cos(\phi_m)+1)}\Big(2(\widehat{\varepsilon}_{\mathcal{P}}\cos(\phi_m)\sin(s^{\pm}-\phi_m)\mathrm{e}^{\mho^{\pm}(x,t)}-\varepsilon_b\sin(s^{\pm}))\\
&\quad\times\cos(\Phi^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})+(\aleph_4+\aleph_3)(\widehat{\varepsilon}_{\mathcal{P}}\cos(\phi_m)\sin(s^{\pm})\mathrm{e}^{\mho^{\pm}(x,t)}-\varepsilon_b\sin(s^{\pm}+\phi_m))\\
&\quad\times\cos(\Phi^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})\mp(\aleph_4-\aleph_3)(\widehat{\varepsilon}_{\mathcal{P}}\cos(\phi_m)\cos(s^{\pm})\mathrm{e}^{\mho^{\pm}(x,t)}-\varepsilon_b\cos(s^{\pm}+\phi_m))\\
&\quad\times\sin(\Phi^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})\Big),\\
\widehat{\mathcal{E}}_{\mathcal{SC}}^{(3)}(x,t) &= -\frac{2\varepsilon_b\widehat{\varepsilon}_{\mathcal{P}}\aleph_4\cos(\phi_m)\mathrm{e}^{\mho^{\pm}(x,t)}}{(\aleph_4^2+2\aleph_4\cos(\phi_m)+1)}\big(2\cos(s^{\pm}-\phi_m)\cos(\Phi^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})+(\aleph_4+\aleph_3)\cos(s^{\pm})\\
&\quad\times\cos(\Phi^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})\pm(\aleph_4-\aleph_3)\sin(s^{\pm})\sin(\Phi^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})\big),\\
\widehat{\mathcal{E}}_{\mathcal{SC}}^{(4)}(x,t) &= -\frac{4\varepsilon_b\widehat{\varepsilon}_{\mathcal{P}}\aleph_4^2\sin(\phi_m)\sin(s^{\pm})\mathrm{e}^{\mho^{\pm}(x,t)}}{(\aleph_4^2+2\aleph_4\cos(\phi_m)+1)^2}\big(((\aleph_4+\aleph_3)\cos(\phi_m)+2)\cos(\Phi^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})\\
&\quad\pm(\aleph_4-\aleph_3)\sin(\phi_m)\sin(\Phi^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})\big),\\
\widehat{\mathcal{E}}_{\mathcal{SC}}^{(5)}(x,t) &= -2\varepsilon_b\widehat{\varepsilon}_{\mathcal{P}}\csc(\phi_m)\sin(s^{\pm})\cos(\Phi^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})\sinh(\mho^{\pm}(x,t)),\\
\widehat{\mathcal{E}}_{\mathcal{SC}}^{(6)}(x,t) &= -\frac{4\sin(\phi_m)\sin(s^{\pm}-\phi_m)}{(1-\mathrm{e}^{2\mho^{\pm}(x,t)})}\cos(\Phi^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})\mathrm{e}^{2\mho^{\pm}(x,t)},\\
\widehat{\mathcal{E}}_{\mathcal{SC}}^{(7)}(x,t) &= 2\varepsilon_b\widehat{\varepsilon}_{\mathcal{P}}\cos(s^{\pm}-\phi_m)\cos(\Phi^{\pm}(z_o,t)\pm(2\mp1)\tfrac{\pi}{4})\mathrm{e}^{\mho^{\pm}(x,t)},
\end{aligned}$$

and $\psi^{\pm}(\cdot)$, $\{\aleph_n\}_{n=1}^{4}$, $\widehat{\varepsilon}_{\mathcal{P}}$, $\mho^{\pm}(x,t)$, *and* $\Phi^{\pm}(z_o,t)$ *given in Theorem* 2.2.1, *Eqs.* (22)–(23), (24), (26), (27)–(28), *and* (30)–(32), *respectively.*

One important application of the asymptotic results obtained in this paper is related to the so-called N-*dark soliton scattering*, namely, the explicit calculation of the nth dark soliton position shift in the presence of the (non-trivial) continuous spectrum. Note that, unlike bright solitons of the focusing NLSE (with rapidly decaying, in the sense of Schwartz, initial data), which undergo both position and phase shifts [7, 12, 52], dark solitons of the D_fNLSE (for the finite-density initial data considered here) only undergo a position shift [13]. This leads to the following (see, also, Corollary 2.2.2)

Corollary 2.2.1. *Set*

$$\Delta\widetilde{x}_n:=\widetilde{x}_n^{+}-\widetilde{x}_n^{-} \qquad \text{and} \qquad \Delta\widehat{x}_n:=\widehat{x}_n^{+}-\widehat{x}_n^{-}, \qquad n\in\{1,2,\dots,N\}.$$

As $t\to\pm\infty$ *and* $x\to\mp\infty$ *such that* $z_o:=x/t<-2$ *and* $(x,t)\in\{(x,t);\ x+2t\cos(\phi_n)=\mathcal{O}(1),\ \phi_n\in(0,\pi)\}$,

$$\Delta\widetilde{x}_n=\sum_{k=1}^{N}\frac{\operatorname{sgn}(n-k)}{\sin(\phi_n)}\ln\left(\left|\frac{\sin(\frac{1}{2}(\phi_n+\phi_k))}{\sin(\frac{1}{2}(\phi_n-\phi_k))}\right|\right)+\left(\int_0^{\lambda_2}+\int_{\lambda_1}^{+\infty}-\int_{-\infty}^{0}-\int_{\lambda_2}^{\lambda_1}\right)\frac{\ln(1-|r(\mu)|^2)}{(\mu^2-2\mu\cos(\phi_n)+1)}\frac{\mathrm{d}\mu}{2\pi},$$

and, as $t\to\pm\infty$ *and* $x\to\pm\infty$ *such that* $z_o>2$ *and* $(x,t)\in\{(x,t);\ x+2t\cos(\phi_n)=\mathcal{O}(1),\ \phi_n\in(-\pi,0)\}$,

$$\Delta\widehat{x}_n=\sum_{k=1}^{N}\frac{\operatorname{sgn}(n-k)}{\sin(\phi_n)}\ln\left(\left|\frac{\sin(\frac{1}{2}(\phi_n+\phi_k))}{\sin(\frac{1}{2}(\phi_n-\phi_k))}\right|\right)+\left(\int_{-\infty}^{\aleph_4}+\int_{\aleph_3}^{0}-\int_{\aleph_4}^{\aleph_3}-\int_{0}^{+\infty}\right)\frac{\ln(1-|r(\mu)|^2)}{(\mu^2+2\mu\cos(\phi_n)+1)}\frac{\mathrm{d}\mu}{2\pi}.$$

Proof. Follows from the definition of $\Delta\widetilde{x}_n$ and $\Delta\widehat{x}_n$, and Theorem 2.2.1, Eqs. (15) and (28). □

Theorem 2.2.3. *For* $r(\zeta)\in\mathcal{S}^1_{\mathbb{C}}(\mathbb{R})$, *let* $m(x,t;\zeta)$ *be the solution of the Riemann-Hilbert problem formulated in Lemma* 2.1.2. *Let* $u(x,t)$, *the solution of the Cauchy problem for the* D_fNLSE *with finite-density initial data* $u(x,0):=u_o(x)=_{x\to\pm\infty}u_o(\pm\infty)(1+o(1))$, *where* $u_o(\pm\infty):=\exp(\frac{\mathrm{i}(1\mp1)\theta}{2})$, $0\leqslant\theta=-2\sum_{n=1}^{N}\sin(\phi_n)-\int_{-\infty}^{+\infty}\frac{\ln(1-|r(\mu)|^2)}{\mu}\frac{\mathrm{d}\mu}{2\pi}<2\pi$, $u_o(x)\in\mathbf{C}^{\infty}(\mathbb{R})$, *and* $u_o(x)-u_o(\pm\infty)\in\mathcal{S}_{\mathbb{C}}(\mathbb{R}_{\pm})$, *be defined by Eq.* (5). *Then, for* $\theta_{\gamma_m}=\varepsilon_b\pi/2$, $\varepsilon_b\in\{\pm1\}$, $m\in\{1,2,\dots,N\}$, *as* $t\to\pm\infty$ *and* $x\to\mp\infty$ *such that* $z_o:=x/t\in(-2,0)$ *and* $(x,t)\in\{(x,t);\ x+2t\cos(\phi_m)=\mathcal{O}(1),\ \phi_m\in(0,\pi)\}$,

$$\begin{aligned}u(x,t)=\mathrm{e}^{-\mathrm{i}(\varkappa^{\pm}(1)+s^{\pm})}\Bigg(&\frac{(1+\varepsilon_b\widetilde{\varepsilon}_{\mathcal{P}}\mathrm{e}^{-2\mathrm{i}\phi_m+\Omega^{\pm}_{\sharp}(x,t)})}{(1+\varepsilon_b\widetilde{\varepsilon}_{\mathcal{P}}\mathrm{e}^{\Omega^{\pm}_{\sharp}(x,t)})}\\&+\mathcal{O}\left(\mathrm{e}^{-4|t|\min\limits_{k\neq m\in\{1,2,\dots,N\}}\{\sin(\phi_k)|\cos(\phi_k)-\cos(\phi_m)|\}}\right)\Bigg),\end{aligned}\tag{46}$$

where $s^{\pm}$ *and* $\widetilde{\varepsilon}_{\mathcal{P}}$, *respectively, are given in Theorem* 2.2.1, *Eqs.* (11) *and* (13),

$$\varkappa^{+}(j)=\int_{-\infty}^{0}\frac{\ln(1-|r(\mu)|^2)}{\mu^j}\frac{\mathrm{d}\mu}{2\pi},\qquad\varkappa^{-}(j)=\int_{0}^{+\infty}\frac{\ln(1-|r(\mu)|^2)}{\mu^j}\frac{\mathrm{d}\mu}{2\pi},\qquad j\in\{0,1\},\tag{47}$$

$$\Omega^{\pm}_{\sharp}(x,t)=-2\sin(\phi_m)(x+2t\cos(\phi_m)-\widetilde{x}^{\pm}_{m,\sharp}),\tag{48}$$

and

$$\begin{aligned}\widetilde{x}^{\pm}_{m,\sharp}=\pm\sum_{k=1}^{N}\frac{\operatorname{sgn}(m-k)}{2\sin(\phi_m)}\ln\left(\left|\frac{\sin(\frac{1}{2}(\phi_m+\phi_k))}{\sin(\frac{1}{2}(\phi_m-\phi_k))}\right|\right)\pm\frac{1}{2}\left(\int_0^{+\infty}-\int_{-\infty}^{0}\right)\frac{\ln(1-|r(\mu)|^2)}{(\mu^2-2\mu\cos(\phi_m)+1)}\frac{\mathrm{d}\mu}{2\pi}\\+\frac{\ln(|\gamma_m|)}{2\sin(\phi_m)},\end{aligned}\tag{49}$$

and, as $t\to\pm\infty$ *and* $x\to\pm\infty$ *such that* $z_o\in(0,2)$ *and* $(x,t)\in\{(x,t);\ x+2t\cos(\phi_m)=\mathcal{O}(1),\ \phi_m\in(-\pi,0)\}$,

$$\begin{aligned}u(x,t)=-\mathrm{e}^{-\mathrm{i}(\varkappa^{\pm}(1)+s^{\pm})}\Bigg(&\frac{(1+\varepsilon_b\widehat{\varepsilon}_{\mathcal{P}}\mathrm{e}^{-2\mathrm{i}\phi_m+\Omega^{\pm}_{\natural}(x,t)})}{(1+\varepsilon_b\widehat{\varepsilon}_{\mathcal{P}}\mathrm{e}^{\Omega^{\pm}_{\natural}(x,t)})}\\&+\mathcal{O}\left(\mathrm{e}^{-4|t|\min\limits_{k\neq m\in\{1,2,\dots,N\}}\{|\sin(\phi_k)||\cos(\phi_k)-\cos(\phi_m)|\}}\right)\Bigg),\end{aligned}\tag{50}$$

where $\widehat{\varepsilon}_{\mathcal{P}}$ *is given in Theorem* 2.2.1, *Eq.* (26),

$$\Omega^{\pm}_{\natural}(x,t)=-2\sin(\phi_m)(x+2t\cos(\phi_m)-\widehat{x}^{\pm}_{m,\natural}),\tag{51}$$

and

$$\begin{aligned}\widehat{x}^{\pm}_{m,\natural}=\pm\sum_{k=1}^{N}\frac{\operatorname{sgn}(m-k)}{2\sin(\phi_m)}\ln\left(\left|\frac{\sin(\frac{1}{2}(\phi_m+\phi_k))}{\sin(\frac{1}{2}(\phi_m-\phi_k))}\right|\right)\pm\frac{1}{2}\left(\int_{-\infty}^{0}-\int_{0}^{+\infty}\right)\frac{\ln(1-|r(\mu)|^2)}{(\mu^2+2\mu\cos(\phi_m)+1)}\frac{\mathrm{d}\mu}{2\pi}\\+\frac{\ln(|\gamma_m|)}{2\sin(\phi_m)}.\end{aligned}\tag{52}$$

Theorem 2.2.4. *For $r(\zeta) \in \mathcal{S}^1_{\mathbb{C}}(\mathbb{R})$, let $m(x,t;\zeta)$ be the solution of the Riemann-Hilbert problem formulated in Lemma 2.1.2. Let $u(x,t)$, the solution of the Cauchy problem for the* D_fNLSE *with finite-density initial data $u(x,0):=u_o(x)=_{x\to\pm\infty} u_o(\pm\infty)(1+o(1))$, where $u_o(\pm\infty):=\exp(\frac{\mathrm{i}(1\mp 1)\theta}{2})$, $0\leqslant\theta=-2\sum_{n=1}^N\sin(\phi_n)-\int_{-\infty}^{+\infty}\frac{\ln(1-|r(\mu)|^2)}{\mu}\frac{\mathrm{d}\mu}{2\pi}<2\pi$, $u_o(x)\in\mathbf{C}^\infty(\mathbb{R})$, and $u_o(x)-u_o(\pm\infty)\in\mathcal{S}_{\mathbb{C}}(\mathbb{R}_\pm)$, be defined by Eq.* (5)*, and $\int_{+\infty}^x(|u(x',t)|^2-1)\,\mathrm{d}x'$ be defined by Eq.* (6)*. Let $\epsilon\in\{\pm1\}$. Then, for $\theta_{\gamma_m}=\varepsilon_b\pi/2$, $\varepsilon_b\in\{\pm1\}$, $m\in\{1,2,\dots,N\}$, as $t\to\pm\infty$ and $x\to\mp\infty$ such that $z_o\in(-2,0)$ and $(x,t)\in\{(x,t);\ x+2t\cos(\phi_m)=\mathcal{O}(1),\ \phi_m\in(0,\pi)\}$,*

$$\int_{\mathrm{sgn}(\epsilon)\infty}^x(|u(x',t)|^2-1)\,\mathrm{d}x'=\widetilde{\mathrm{S}}^\pm_\epsilon+\mathcal{H}^\pm_{\sharp,\epsilon}+\frac{2\varepsilon_b\widetilde{\varepsilon}_{\mathcal{P}}\sin(\phi_m)\mathrm{e}^{\Omega^\pm_\sharp(x,t)}}{(1+\varepsilon_b\widetilde{\varepsilon}_{\mathcal{P}}\mathrm{e}^{\Omega^\pm_\sharp(x,t)})}+\mathcal{O}\left(\mathrm{e}^{-4|t|\min\limits_{k\neq m\in\{1,2,\dots,N\}}\{\sin(\phi_k)|\cos(\phi_k)-\cos(\phi_m)|\}}\right),\tag{53}$$

where $\widetilde{\varepsilon}_{\mathcal{P}}$ is given in Theorem 2.2.1*, Eq.* (13)*, $\widetilde{\mathrm{S}}^\pm_\epsilon$ are given in Theorem* 2.2.2*, Eq.* (35)*, $\Omega^\pm_\sharp(x,t)$ are given in Theorem* 2.2.3*, Eqs.* (48)–(49)*,*

$$\mathcal{H}^+_{\sharp,\epsilon}=\begin{cases}\varkappa^+(0), & \epsilon=+1,\\ -\varkappa^-(0), & \epsilon=-1,\end{cases}\qquad \mathcal{H}^-_{\sharp,\epsilon}=\begin{cases}\varkappa^-(0), & \epsilon=+1,\\ -\varkappa^+(0), & \epsilon=-1,\end{cases}\tag{54}$$

and $\varkappa^\pm(\cdot)$ are given in Theorem 2.2.3*, Eq.* (47)*, and, as $t\to\pm\infty$ and $x\to\pm\infty$ such that $z_o\in(0,2)$ and $(x,t)\in\{(x,t);\ x+2t\cos(\phi_m)=\mathcal{O}(1),\ \phi_m\in(-\pi,0)\}$,*

$$\int_{\mathrm{sgn}(\epsilon)\infty}^x(|u(x',t)|^2-1)\,\mathrm{d}x'=\widehat{\mathrm{S}}^\pm_\epsilon+\mathcal{H}^\pm_{\natural,\epsilon}+\frac{2\varepsilon_b\widehat{\varepsilon}_{\mathcal{P}}\sin(\phi_m)\mathrm{e}^{\Omega^\pm_\natural(x,t)}}{(1+\varepsilon_b\widehat{\varepsilon}_{\mathcal{P}}\mathrm{e}^{\Omega^\pm_\natural(x,t)})}+\mathcal{O}\left(\mathrm{e}^{-4|t|\min\limits_{k\neq m\in\{1,2,\dots,N\}}\{|\sin(\phi_k)||\cos(\phi_k)-\cos(\phi_m)|\}}\right),\tag{55}$$

where $\widehat{\varepsilon}_{\mathcal{P}}$ is given in Theorem 2.2.1*, Eq.* (26)*, $\widehat{\mathrm{S}}^\pm_\epsilon$ are given in Theorem* 2.2.2*, Eq.* (41)*, $\Omega^\pm_\natural(x,t)$ are given in Theorem* 2.2.3*, Eqs.* (51)–(52)*, and*

$$\mathcal{H}^+_{\natural,\epsilon}=\begin{cases}\varkappa^-(0), & \epsilon=+1,\\ -\varkappa^+(0), & \epsilon=-1,\end{cases}\qquad \mathcal{H}^-_{\natural,\epsilon}=\begin{cases}\varkappa^+(0), & \epsilon=+1,\\ -\varkappa^-(0), & \epsilon=-1.\end{cases}\tag{56}$$

Corollary 2.2.2. *Set*

$$\Delta x^\sharp_n:=\widetilde{x}^+_{n,\sharp}-\widetilde{x}^-_{n,\sharp}\qquad\text{and}\qquad\Delta x^\natural_n:=\widehat{x}^+_{n,\natural}-\widehat{x}^-_{n,\natural},\qquad n\in\{1,2,\dots,N\}.$$

As $t\to\pm\infty$ and $x\to\mp\infty$ such that $z_o:=x/t\in(-2,0)$ and $(x,t)\in\{(x,t);\ x+2t\cos(\phi_n)=\mathcal{O}(1),\ \phi_n\in(0,\pi)\}$,

$$\Delta x^\sharp_n=\sum_{k=1}^N\frac{\mathrm{sgn}(n-k)}{\sin(\phi_n)}\ln\left(\left|\frac{\sin(\frac{1}{2}(\phi_n+\phi_k))}{\sin(\frac{1}{2}(\phi_n-\phi_k))}\right|\right)+\left(\int_0^{+\infty}-\int_{-\infty}^0\right)\frac{\ln(1-|r(\mu)|^2)}{(\mu^2-2\mu\cos(\phi_n)+1)}\frac{\mathrm{d}\mu}{2\pi},$$

and, as $t\to\pm\infty$ and $x\to\pm\infty$ such that $z_o\in(0,2)$ and $(x,t)\in\{(x,t);\ x+2t\cos(\phi_n)=\mathcal{O}(1),\ \phi_n\in(-\pi,0)\}$,

$$\Delta x^\natural_n=\sum_{k=1}^N\frac{\mathrm{sgn}(n-k)}{\sin(\phi_n)}\ln\left(\left|\frac{\sin(\frac{1}{2}(\phi_n+\phi_k))}{\sin(\frac{1}{2}(\phi_n-\phi_k))}\right|\right)+\left(\int_{-\infty}^0-\int_0^{+\infty}\right)\frac{\ln(1-|r(\mu)|^2)}{(\mu^2+2\mu\cos(\phi_n)+1)}\frac{\mathrm{d}\mu}{2\pi}.$$

Proof. Follows from the definition of $\Delta x^\sharp_n$ and $\Delta x^\natural_n$, and Theorem 2.2.3, Eqs. (49) and (52). □

Remark 2.2.3. In this paper, the complete details of the asymptotic analysis are presented for the case $t\to+\infty$ and $x\to-\infty$ such that $z_o:=x/t<-2$ and $(x,t)\in\{(x,t);\ x+2t\cos(\phi_n)=\mathcal{O}(1),\ \phi_n\in(0,\pi)\}$, and the final results for the analogous asymptotic analysis as $t\to-\infty$ and $x\to+\infty$ such that $z_o<-2$ and $(x,t)\in\{(x,t);\ x+2t\cos(\phi_n)=\mathcal{O}(1),\ \phi_n\in(0,\pi)\}$ are given in Appendix A. The remaining cases are treated similarly, and one uses the results of Appendix B to obtain the corresponding leading-order asymptotic expansions.

3 The Model RHP

In this section, the RHP studied asymptotically (as $t \to +\infty$) in Section 4, the so-called model RHP, is derived: it is obtained from the (normalised at ∞) RHP for $m(x,t;\zeta)$ formulated in Lemma 2.1.2 via an ingenius method due to Deift *et al.* [34] (see below). Set $\urcorner_m := \{(x,t);\ x+2t\cos(\phi_m) = \mathcal{O}(1),\ \phi_m \in (0,\pi)\}$, $m \in \{1,2,\dots,N\}$: note that the mth dark soliton "trajectory" in the (x,t)-plane, $\mathbb{R}^2$, belongs to $\urcorner_m$. From Lemma 2.1.2, *(iii)*, and the dark soliton ordering adopted in Subsection 2.2, one notes that, as $t \to +\infty$ and $x \to -\infty$ such that $z_o := x/t < -2$ and $(x,t) \in \urcorner_m$: (1) for $n=m$, $g_n(x,t)\restriction_{\urcorner_m} = \mathcal{O}(1)$; (2) for $n<m$, $g_n(x,t)\restriction_{\urcorner_m} = \mathcal{O}(\exp(-4t\sin(\phi_n)|\cos(\phi_n)-\cos(\phi_m)|)) \to 0$; and (3) for $n>m$, $g_n(x,t)\restriction_{\urcorner_m} = \mathcal{O}(\exp(4t\sin(\phi_n)|\cos(\phi_n)-\cos(\phi_m)|)) \to \infty$. Thus (cf. Remark 2.1.3), since the RHP for $m(x,t;\zeta)$ formulated in Lemma 2.1.2 is asymptotically solvable for the (x,t)-sector stated above, one deduces that, along the trajectory of the (arbitrarily fixed) mth dark soliton: (1) for $n=m$, $\mathrm{Res}(m(x,t;\zeta);\varsigma_n) = \left(\begin{smallmatrix} \mathcal{O}(1) & 0 \\ \mathcal{O}(1) & 0 \end{smallmatrix}\right)$ and $\mathrm{Res}(m(x,t;\zeta);\overline{\varsigma_n}) = \left(\begin{smallmatrix} 0 & \mathcal{O}(1) \\ 0 & \mathcal{O}(1) \end{smallmatrix}\right)$; (2) for $n<m$, $\mathrm{Res}(m(x,t;\zeta);\varsigma_n) = \left(\begin{smallmatrix} \mathcal{O}(\blacklozenge) & 0 \\ \mathcal{O}(\blacklozenge) & 0 \end{smallmatrix}\right) \to \mathbf{0}$ and $\mathrm{Res}(m(x,t;\zeta);\overline{\varsigma_n}) = \left(\begin{smallmatrix} 0 & \mathcal{O}(\blacklozenge) \\ 0 & \mathcal{O}(\blacklozenge) \end{smallmatrix}\right) \to \mathbf{0}$, where $\blacklozenge := \exp(-4t\sin(\phi_n)|\cos(\phi_n)-\cos(\phi_m)|)$; and (3) for $n>m$, $\mathrm{Res}(m(x,t;\zeta);\varsigma_n) = \left(\begin{smallmatrix} \mathcal{O}(\blacklozenge^{-1}) & 0 \\ \mathcal{O}(\blacklozenge^{-1}) & 0 \end{smallmatrix}\right) \to \left(\begin{smallmatrix} \infty & 0 \\ \infty & 0 \end{smallmatrix}\right)$ and $\mathrm{Res}(m(x,t;\zeta);\overline{\varsigma_n}) = \left(\begin{smallmatrix} 0 & \mathcal{O}(\blacklozenge^{-1}) \\ 0 & \mathcal{O}(\blacklozenge^{-1}) \end{smallmatrix}\right) \to \left(\begin{smallmatrix} 0 & \infty \\ 0 & \infty \end{smallmatrix}\right)$. Hence, along the trajectory of the (arbitrarily fixed) mth dark soliton, there are exponentially growing polar (residue) conditions for solitons n with $n \in \{m+1, m+2, \dots, N\}$. In a paper dealing with the Toda Rarefaction Problem [34], Deift *et al.* showed how this problem could be dealt with. Proceeding from the construction of Zhou [44, 45, 46] related to the singular RHP (see the synopsis below Theorem C.1.4 in Appendix C), one uses the method of Deift *et al.* to "replace" the poles which give rise to the exponentially growing residue conditions by jump matrices on mutually disjoint, and disjoint with respect to σ_c, "small" circles (see [46], Section 2, Remark 2.18, for a discussion about the radii of these circles) in such a way that the jump matrices on these small circles behave like I + exponentially decreasing terms (as $t \to +\infty$), thus constructing the *augmented* contour $\sigma_{\text{augmented}} := \sigma_c \cup (\cup_{n=m+1}^{N} \partial(\text{small circles}))$. Thus, instead of the original RHP, one obtains an augmented (and normalised at ∞) RHP with $2(N-m)$ fewer poles and $2(N-m)$ additional circles with jump conditions stated on them. Finally, by "removing" the $2(N-m)$ small circles from the augmented RHP, one arrives at an asymptotically solvable, equivalent, "model" RHP, in the sense that a solution of the equivalent RHP gives a solution of the augmented RHP and *vice versa*; in particular, if there are two RHPs, $(\mathcal{X}_1(\lambda), \upsilon_1(\lambda), \Gamma_1)$ and $(\mathcal{X}_2(\lambda), \upsilon_2(\lambda), \Gamma_2)$, say, with $\Gamma_2 \subset \Gamma_1$ and $\upsilon_1(\lambda)\restriction_{\Gamma_1 \setminus \Gamma_2} =_{t \to +\infty} \mathrm{I} + o(1)$, then, modulo $o(1)$ estimates, their solutions, $\mathcal{X}_1(\lambda)$ and $\mathcal{X}_2(\lambda)$, respectively, are asymptotically equal. Actually, as will be shown below (see Lemma 3.5), the solution of the model RHP approximates, up to terms that are exponentially small (as $t \to +\infty$), the solution of the augmented RHP (hence the original RHP).

The reason for introducing the factor $\delta(\zeta)$ in Lemma 3.1 below is given in Section 4 of [38].

Remark 3.1. For notational convenience, all explicit x,t dependencies are hereafter suppressed, except where absolutely necessary and/or where confusion may arise.

Lemma 3.1. *For $r(\zeta) \in \mathcal{S}^1_{\mathbb{C}}(\mathbb{R})$, let $m(\zeta)\colon \mathbb{C} \setminus (\sigma_d \cup \sigma_c) \to \mathrm{M}_2(\mathbb{C})$ be the solution of the* RHP *formulated in Lemma* 2.1.2*. Set*

$$\widehat{m}(\zeta) := m(\zeta)(\delta(\zeta))^{-\sigma_3},$$

where $\delta(\zeta) = \exp\left(\left(\int_{-\infty}^{0} + \int_{\lambda_2}^{\lambda_1}\right) \frac{\ln(1-|r(\mu)|^2)}{(\mu-\zeta)} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}}\right)$, *with* λ_1 *and* λ_2 *given in Theorem* 2.2.1*, Eq.* (10)*,* $\delta(\zeta)\overline{\delta(\overline{\zeta})} = 1$*,* $\delta(\zeta)\delta(\zeta^{-1}) = \delta(0)$*, and* $||(\delta(\cdot))^{\pm 1}||_{\mathcal{L}^\infty(\mathbb{C})} := \sup_{\zeta \in \mathbb{C}} |(\delta(\zeta))^{\pm 1}| < \infty$*. Then* $\widehat{m}(\zeta)\colon \mathbb{C} \setminus (\sigma_d \cup \sigma_c) \to \mathrm{M}_2(\mathbb{C})$ *solves the following* RHP:

(i) $\widehat{m}(\zeta)$ *is piecewise (sectionally) meromorphic* $\forall\, \zeta \in \mathbb{C} \setminus \sigma_c$;

(ii) $\widehat{m}_\pm(\zeta) := \lim_{\substack{\zeta' \to \zeta \\ \pm\mathrm{Im}(\zeta')>0}} \widehat{m}(\zeta')$ *satisfy the jump condition*

$$\widehat{m}_+(\zeta) = \widehat{m}_-(\zeta)\exp(-\mathrm{i}k(\zeta)(x+2\lambda(\zeta)t)\mathrm{ad}(\sigma_3))\widehat{\mathcal{G}}(\zeta), \quad \zeta \in \mathbb{R},$$

where

$$\widehat{\mathcal{G}}(\zeta) = \begin{pmatrix} (1-r(\zeta)\overline{r(\overline{\zeta})})\delta_-(\zeta)(\delta_+(\zeta))^{-1} & -\overline{r(\overline{\zeta})}\,\delta_-(\zeta)\delta_+(\zeta) \\ r(\zeta)(\delta_-(\zeta)\delta_+(\zeta))^{-1} & (\delta_-(\zeta))^{-1}\delta_+(\zeta) \end{pmatrix};$$

(iii) $\widehat{m}(\zeta)$ *has simple poles in* $\sigma_d = \cup_{n=1}^N(\{\varsigma_n\}\cup\{\overline{\varsigma_n}\})$ *with*

$$\begin{aligned}
&\mathrm{Res}(\widehat{m}(\zeta);\varsigma_n)=\lim_{\zeta\to\varsigma_n}\widehat{m}(\zeta)g_n(\delta(\varsigma_n))^{-2}\sigma_-, && n\in\{1,2,\dots,N\},\\
&\mathrm{Res}(\widehat{m}(\zeta);\overline{\varsigma_n})=\sigma_1\overline{\mathrm{Res}(\widehat{m}(\zeta);\varsigma_n)}\,\sigma_1, && n\in\{1,2,\dots,N\},
\end{aligned}$$

where $g_n := |g_n|\mathrm{e}^{\mathrm{i}\theta_{g_n}}\exp(2\mathrm{i}k(\varsigma_n)(x+2\lambda(\varsigma_n)t))$*, with*

$$|g_n| = 2|\gamma_n|\sin(\phi_n)\exp\left(\int_{-\infty}^{+\infty}\frac{\sin(\phi_n)\ln(1-|r(\mu)|^2)}{(\mu^2-2\mu\cos(\phi_n)+1)}\frac{\mathrm{d}\mu}{2\pi}\right)\prod_{\substack{k=1\\k\neq n}}^N\frac{\sin(\frac{1}{2}(\phi_n+\phi_k))}{\sin(\frac{1}{2}(\phi_n-\phi_k))},$$

$$\theta_{g_n} = \theta_{\gamma_n}+\frac{\pi}{2}-\frac{\theta}{2}-\int_{-\infty}^{+\infty}\frac{(\mu-\cos\phi_n)\ln(1-|r(\mu)|^2)}{(\mu^2-2\mu\cos(\phi_n)+1)}\frac{\mathrm{d}\mu}{2\pi}-\sum_{\substack{k=1\\k\neq n}}^N\phi_k, \qquad \theta_{\gamma_n}=\pm\tfrac{\pi}{2};$$

(iv) $\det(\widehat{m}(\zeta))|_{\zeta=\pm1}=0$;

(v) $\widehat{m}(\zeta) \underset{\zeta\to 0}{=} \zeta^{-1}(\delta(0))^{\sigma_3}\sigma_2+\mathcal{O}(1)$;

(vi) $\widehat{m}(\zeta) \underset{\substack{\zeta\to\infty\\ \zeta\in\mathbb{C}\setminus(\sigma_d\cup\sigma_c)}}{=} \mathrm{I}+\mathcal{O}(\zeta^{-1})$;

(vii) $\widehat{m}(\zeta)=\sigma_1\overline{\widehat{m}(\overline{\zeta})}\,\sigma_1$ *and* $\widehat{m}(\zeta^{-1})=\zeta\widehat{m}(\zeta)(\delta(0))^{\sigma_3}\sigma_2$.

Let

$$u(x,t) := \mathrm{i}\lim_{\substack{\zeta\to\infty\\ \zeta\in\mathbb{C}\setminus(\sigma_d\cup\sigma_c)}}(\zeta(\widehat{m}(\zeta)(\delta(\zeta))^{\sigma_3}-\mathrm{I}))_{12}, \tag{57}$$

and

$$\int_{+\infty}^{x}(|u(x',t)|^2-1)\,\mathrm{d}x' := -\mathrm{i}\lim_{\substack{\zeta\to\infty\\ \zeta\in\mathbb{C}\setminus(\sigma_d\cup\sigma_c)}}(\zeta(\widehat{m}(\zeta)(\delta(\zeta))^{\sigma_3}-\mathrm{I}))_{11}. \tag{58}$$

Then $u(x,t)$ *is the solution of the Cauchy problem for the* D_fNLSE.

Proof. The RHP for $\widehat{m}(\zeta)$ (respectively, Eqs. (57) and (58)) follows from the RHP for $m(\zeta)$ formulated in Lemma 2.1.2 (respectively, Eqs. (5) and (6)) upon using $\widehat{m}(\zeta):=m(\zeta)(\delta(\zeta))^{-\sigma_3}$, with $\delta(\zeta)$ given in the Lemma. □

Definition 3.1. *For* $m\in\{1,2,\dots,N\}$ *and* $\{\varsigma_n\}_{n=m+1}^N\subset\mathbb{C}_+$ *(respectively,* $\{\overline{\varsigma_n}\}_{n=m+1}^N\subset\mathbb{C}_-$*), define the clockwise (respectively, counter-clockwise) oriented circles* $\widehat{\mathcal{K}}_n:=\{\zeta;\,|\zeta-\varsigma_n|=\widehat{\varepsilon}_n^{\mathcal{K}}\}$ *(respectively,* $\widehat{\mathcal{L}}_n:=\{\zeta;\,|\zeta-\overline{\varsigma_n}|=\widehat{\varepsilon}_n^{\mathcal{L}}\}$*), with* $\widehat{\varepsilon}_n^{\mathcal{K}}$ *(respectively,* $\widehat{\varepsilon}_n^{\mathcal{L}}$*) chosen sufficiently small such that* $\widehat{\mathcal{K}}_n\cap\widehat{\mathcal{K}}_{n'}=\widehat{\mathcal{L}}_n\cap\widehat{\mathcal{L}}_{n'}=\widehat{\mathcal{K}}_n\cap\widehat{\mathcal{L}}_n=\widehat{\mathcal{K}}_n\cap\sigma_c=\widehat{\mathcal{L}}_n\cap\sigma_c=\emptyset\ \forall\ n\neq n'\in\{m+1,m+2,\dots,N\}$.

Remark 3.2. Note that the orientation for $\widehat{\mathcal{K}}_n$ $(\subset\mathbb{C}_+)$ and $\widehat{\mathcal{L}}_n$ $(\subset\mathbb{C}_-)$ is consistent with Eq. (C.1) (see Appendix C).

Lemma 3.2. *For* $r(\zeta)\in\mathcal{S}_{\mathbb{C}}^1(\mathbb{R})$*, let* $\widehat{m}(\zeta)\colon\mathbb{C}\setminus(\sigma_d\cup\sigma_c)\to\mathrm{M}_2(\mathbb{C})$ *be the solution of the* RHP *formulated in Lemma* 3.1. *Set*

$$\widehat{m}^{\flat}(\zeta):=\begin{cases}\widehat{m}(\zeta), & \zeta\in\mathbb{C}\setminus(\sigma_c\cup(\cup_{n=m+1}^N(\widehat{\mathcal{K}}_n\cup\mathrm{int}(\widehat{\mathcal{K}}_n)\cup\widehat{\mathcal{L}}_n\cup\mathrm{int}(\widehat{\mathcal{L}}_n)))),\\ \widehat{m}(\zeta)\left(\mathrm{I}-\frac{g_n(\delta(\varsigma_n))^{-2}}{(\zeta-\varsigma_n)}\sigma_-\right), & \zeta\in\mathrm{int}(\widehat{\mathcal{K}}_n), \quad n\in\{m+1,m+2,\dots,N\},\\ \widehat{m}(\zeta)\left(\mathrm{I}+\frac{\overline{g_n(\delta(\varsigma_n))^{-2}}}{(\zeta-\overline{\varsigma_n})}\sigma_+\right), & \zeta\in\mathrm{int}(\widehat{\mathcal{L}}_n), \quad n\in\{m+1,m+2,\dots,N\}.\end{cases}$$

Then $\widehat{m}^{\flat}(\zeta)\colon\mathbb{C}\setminus((\sigma_d\setminus\cup_{n=m+1}^N(\{\varsigma_n\}\cup\{\overline{\varsigma_n}\}))\cup(\sigma_c\cup(\cup_{n=m+1}^N(\widehat{\mathcal{K}}_n\cup\widehat{\mathcal{L}}_n))))\to\mathrm{M}_2(\mathbb{C})$ *solves the following* RHP:

(i) $\widehat{m}^{\flat}(\zeta)$ *is piecewise (sectionally) meromorphic* $\forall\,\zeta\in\mathbb{C}\setminus(\sigma_c\cup(\cup_{n=m+1}^N(\widehat{\mathcal{K}}_n\cup\widehat{\mathcal{L}}_n)))$;

(ii) $\widehat{m}^{\flat}_{\pm}(\zeta):=\lim_{\substack{\zeta'\to\zeta \\ \zeta'\in\pm\,\text{side of}\,\sigma_c\cup(\cup_{n=m+1}^{N}(\widehat{\mathcal{K}}_n\cup\widehat{\mathcal{L}}_n))}}\widehat{m}^{\flat}(\zeta')$ *satisfy the jump condition*

$$\widehat{m}^{\flat}_{+}(\zeta)=\widehat{m}^{\flat}_{-}(\zeta)\widehat{v}^{\flat}(\zeta),\quad \zeta\in\sigma_c\cup(\cup_{n=m+1}^{N}(\widehat{\mathcal{K}}_n\cup\widehat{\mathcal{L}}_n)),$$

where

$$\widehat{v}^{\flat}(\zeta)=\begin{cases}\exp(-\mathrm{i}k(\zeta)(x+2\lambda(\zeta)t)\mathrm{ad}(\sigma_3))\widehat{\mathcal{G}}(\zeta), & \zeta\in\mathbb{R},\\ \mathrm{I}+\frac{g_n(\delta(\varsigma_n))^{-2}}{(\zeta-\varsigma_n)}\sigma_-, & \zeta\in\widehat{\mathcal{K}}_n,\quad n\in\{m+1,m+2,\dots,N\},\\ \mathrm{I}+\frac{\overline{g_n(\delta(\varsigma_n))^{-2}}}{(\zeta-\overline{\varsigma_n})}\sigma_+, & \zeta\in\widehat{\mathcal{L}}_n,\quad n\in\{m+1,m+2,\dots,N\},\end{cases}$$

with $\widehat{\mathcal{G}}(\zeta)$ *given in Lemma* 3.1, *(ii);*

(iii) $\widehat{m}^{\flat}(\zeta)$ *has simple poles in* $\sigma_d\setminus\cup_{n=m+1}^{N}(\{\varsigma_n\}\cup\{\overline{\varsigma_n}\})$ *with*

$$\begin{aligned}\mathrm{Res}(\widehat{m}^{\flat}(\zeta);\varsigma_n)&=\lim_{\zeta\to\varsigma_n}\widehat{m}^{\flat}(\zeta)g_n(\delta(\varsigma_n))^{-2}\sigma_-, && n\in\{1,2,\dots,m\},\\ \mathrm{Res}(\widehat{m}^{\flat}(\zeta);\overline{\varsigma_n})&=\sigma_1\overline{\mathrm{Res}(\widehat{m}^{\flat}(\zeta);\varsigma_n)}\,\sigma_1, && n\in\{1,2,\dots,m\};\end{aligned}$$

(iv) $\det(\widehat{m}^{\flat}(\zeta))|_{\zeta=\pm1}=0$;

(v) $\widehat{m}^{\flat}(\zeta)\underset{\zeta\to0}{=}\zeta^{-1}(\delta(0))^{\sigma_3}\sigma_2+\mathcal{O}(1)$;

(vi) as $\zeta\to\infty$, $\zeta\in\mathbb{C}\setminus((\sigma_d\setminus\cup_{n=m+1}^{N}(\{\varsigma_n\}\cup\{\overline{\varsigma_n}\}))\cup(\sigma_c\cup(\cup_{n=m+1}^{N}(\widehat{\mathcal{K}}_n\cup\widehat{\mathcal{L}}_n))))$, $\widehat{m}^{\flat}(\zeta)=\mathrm{I}+\mathcal{O}(\zeta^{-1})$;

(vii) $\widehat{m}^{\flat}(\zeta)=\sigma_1\overline{\widehat{m}^{\flat}(\overline{\zeta})}\,\sigma_1$ *and* $\widehat{m}^{\flat}(\zeta^{-1})=\zeta\widehat{m}^{\flat}(\zeta)(\delta(0))^{\sigma_3}\sigma_2$.

For $\zeta\in\mathbb{C}\setminus((\sigma_d\setminus\cup_{n=m+1}^{N}(\{\varsigma_n\}\cup\{\overline{\varsigma_n}\}))\cup(\sigma_c\cup(\cup_{n=m+1}^{N}(\widehat{\mathcal{K}}_n\cup\widehat{\mathcal{L}}_n))))$, *let*

$$u(x,t):=\mathrm{i}\lim_{\zeta\to\infty}(\zeta(\widehat{m}^{\flat}(\zeta)(\delta(\zeta))^{\sigma_3}-\mathrm{I}))_{12},\tag{59}$$

and

$$\int_{+\infty}^{x}(|u(x',t)|^2-1)\,\mathrm{d}x':=-\mathrm{i}\lim_{\zeta\to\infty}(\zeta(\widehat{m}(\zeta)(\delta(\zeta))^{\sigma_3}-\mathrm{I}))_{11}.\tag{60}$$

Then $u(x,t)$ *is the solution of the Cauchy problem for the* D_fNLSE.

Proof. The RHP for $\widehat{m}^{\flat}(\zeta)$ (respectively, Eqs. (59) and (60)) follows from the RHP for $\widehat{m}(\zeta)$ formulated in Lemma 3.1 (respectively, Eqs. (57) and (58)) upon using the definition of $\widehat{m}^{\flat}(\zeta)$ in terms of $\widehat{m}(\zeta)$ given in the Lemma. □

Remark 3.3. Even though the set (of first-order poles) $\cup_{n=m+1}^{N}(\{\varsigma_n\}\cup\{\overline{\varsigma_n}\})$, giving rise to the exponentially growing residue conditions, has been removed from the specification of the RHP and replaced by jump matrices on $\cup_{n=m+1}^{N}(\widehat{\mathcal{K}}_n\cup\widehat{\mathcal{L}}_n)$, it should be noted that these jump matrices are also exponentially growing (as $t\to+\infty$). These lower/upper diagonal, exponentially growing jump matrices are now replaced, via a finite sequence of transformations, by upper/lower diagonal jump matrices which converge to I as $t\to+\infty$.

Lemma 3.3. *For* $m\in\{1,2,\dots,N\}$, *let* $\sigma_d':=\sigma_d\setminus\cup_{n=m+1}^{n}(\{\varsigma_n\}\cup\{\overline{\varsigma_n}\})$, $\sigma_c':=\sigma_c\cup(\cup_{n=m+1}^{N}(\widehat{\mathcal{K}}_n\cup\widehat{\mathcal{L}}_n))$, *where* $\widehat{\mathcal{K}}_n$ *and* $\widehat{\mathcal{L}}_n$ *are given in Definition* 3.1, *and* $\sigma_{\mathcal{OD}}':=\sigma_d'\cup\sigma_c'$ $(\sigma_d'\cap\sigma_c'=\emptyset)$. *Set*

$$\widehat{m}^{\sharp}(\zeta):=\begin{cases}\widehat{m}^{\flat}(\zeta)\prod_{k=m+1}^{N}(d_k^{+}(\zeta))^{-\sigma_3}, & \zeta\in\mathbb{C}\setminus(\sigma_c'\cup(\cup_{n=m+1}^{N}(\mathrm{int}(\widehat{\mathcal{K}}_n)\cup\mathrm{int}(\widehat{\mathcal{L}}_n)))),\\ \widehat{m}^{\flat}(\zeta)(J_{\widehat{\mathcal{K}}_n}(\zeta))^{-1}\prod_{k=m+1}^{N}(d_k^{-}(\zeta))^{-\sigma_3}, & \zeta\in\mathrm{int}(\widehat{\mathcal{K}}_n),\quad n\in\{m+1,m+2,\dots,N\},\\ \widehat{m}^{\flat}(\zeta)(J_{\widehat{\mathcal{L}}_n}(\zeta))^{-1}\prod_{k=m+1}^{N}(d_k^{-}(\zeta))^{-\sigma_3}, & \zeta\in\mathrm{int}(\widehat{\mathcal{L}}_n),\quad n\in\{m+1,m+2,\dots,N\},\end{cases}$$

where

$$d_n^{+}(\zeta)=\frac{\zeta-\overline{\varsigma_n}}{\zeta-\varsigma_n},\quad \zeta\in\mathbb{C}\setminus(\sigma_c'\cup(\cup_{n=m+1}^{N}(\mathrm{int}(\widehat{\mathcal{K}}_n)\cup\mathrm{int}(\widehat{\mathcal{L}}_n)))),$$

$$d_n^{-}(\zeta)=\begin{cases}\zeta-\overline{\varsigma_n}, & \zeta\in\mathrm{int}(\widehat{\mathcal{K}}_n),\\ (\zeta-\varsigma_n)^{-1}, & \zeta\in\mathrm{int}(\widehat{\mathcal{L}}_n),\end{cases}$$

$J_{\widehat{\mathcal{K}}_n}(\zeta)$ $(\in \mathrm{SL}(2,\mathbb{C}))$ *and* $J_{\widehat{\mathcal{L}}_n}(\zeta)$ $(\in \mathrm{SL}(2,\mathbb{C}))$*, respectively, are holomorphic in* $\cup_{k=m+1}^{N}\mathrm{int}(\widehat{\mathcal{K}}_k)$ *and* $\cup_{l=m+1}^{N}\mathrm{int}(\widehat{\mathcal{L}}_l)$*, with*

$$J_{\widehat{\mathcal{K}}_n}(\zeta)=\begin{pmatrix} \frac{\prod_{\substack{k=m+1\\k\neq n}}^{N}\frac{d_k^+(\zeta)}{d_k^-(\zeta)}-\frac{C_n^{\mathcal{K}}g_n(\delta(\varsigma_n))^{-2}}{(\zeta-\overline{\varsigma_n})^2}\prod_{\substack{k=m+1\\k\neq n}}^{N}\frac{(d_k^+(\zeta))^{-1}}{d_k^-(\zeta)}}{(\zeta-\varsigma_n)} & \frac{C_n^{\mathcal{K}}}{(\zeta-\overline{\varsigma_n})^2}\prod_{\substack{k=m+1\\k\neq n}}^{N}\frac{(d_k^+(\zeta))^{-1}}{d_k^-(\zeta)} \\ -g_n(\delta(\varsigma_n))^{-2}\prod_{\substack{k=m+1\\k\neq n}}^{N}\frac{d_k^-(\zeta)}{d_k^+(\zeta)} & (\zeta-\varsigma_n)\prod_{\substack{k=m+1\\k\neq n}}^{N}\frac{d_k^-(\zeta)}{d_k^+(\zeta)} \end{pmatrix},$$

$$J_{\widehat{\mathcal{L}}_n}(\zeta)=\begin{pmatrix} (\zeta-\overline{\varsigma_n})\prod_{\substack{k=m+1\\k\neq n}}^{N}\frac{d_k^+(\zeta)}{d_k^-(\zeta)} & \overline{g_n(\delta(\varsigma_n))^{-2}}\prod_{\substack{k=m+1\\k\neq n}}^{N}\frac{d_k^+(\zeta)}{d_k^-(\zeta)} \\ -\frac{C_n^{\mathcal{L}}}{(\zeta-\varsigma_n)^2}\prod_{\substack{k=m+1\\k\neq n}}^{N}\frac{d_k^-(\zeta)}{(d_k^+(\zeta))^{-1}} & \frac{\prod_{\substack{k=m+1\\k\neq n}}^{N}\frac{d_k^-(\zeta)}{d_k^+(\zeta)}-\frac{C_n^{\mathcal{L}}\overline{g_n(\delta(\varsigma_n))^{-2}}}{(\zeta-\varsigma_n)^2}\prod_{\substack{k=m+1\\k\neq n}}^{N}\frac{d_k^-(\zeta)}{(d_k^+(\zeta))^{-1}}}{(\zeta-\overline{\varsigma_n})} \end{pmatrix},$$

and

$$C_n^{\mathcal{K}}=\overline{C_n^{\mathcal{L}}}=-4\sin^2(\phi_n)(g_n)^{-1}(\delta(\varsigma_n))^2\,\mathrm{e}^{-2\mathrm{i}\sum_{\substack{j=m+1\\j\neq n}}^{N}\phi_j}\prod_{\substack{k=m+1\\k\neq n}}^{N}\left(\frac{\sin(\frac{1}{2}(\phi_n+\phi_k))}{\sin(\frac{1}{2}(\phi_n-\phi_k))}\right)^2.$$

Then $\widehat{m}^{\sharp}(\zeta)\colon \mathbb{C}\setminus\sigma'_{\mathcal{O}\mathcal{D}}\to \mathrm{M}_2(\mathbb{C})$ *solves the following (augmented)* RHP:

(i) $\widehat{m}^{\sharp}(\zeta)$ *is piecewise (sectionally) meromorphic* $\forall\,\zeta\in\mathbb{C}\setminus\sigma'_c$;

(ii) $\widehat{m}^{\sharp}_{\pm}(\zeta):=\lim_{\substack{\zeta'\to\zeta\\ \zeta'\in\pm\text{ side of }\sigma'_{\mathcal{O}\mathcal{D}}}}\widehat{m}^{\sharp}(\zeta')$ *satisfy the following jump conditions,*

$$\widehat{m}^{\sharp}_{+}(\zeta)=\widehat{m}^{\sharp}_{-}(\zeta)\exp(-\mathrm{i}k(\zeta)(x+2\lambda(\zeta)t)\mathrm{ad}(\sigma_3))\widehat{\mathcal{G}}^{\sharp}(\zeta),\quad \zeta\in\mathbb{R},$$

where

$$\widehat{\mathcal{G}}^{\sharp}(\zeta)=\begin{pmatrix} (1-r(\zeta)\overline{r(\overline{\zeta})})\delta_-(\zeta)(\delta_+(\zeta))^{-1} & -\overline{r(\overline{\zeta})}\,\delta_-(\zeta)\delta_+(\zeta)\prod_{k=m+1}^{N}(d_k^+(\zeta))^2 \\ r(\zeta)(\delta_-(\zeta)\delta_+(\zeta))^{-1}\prod_{k=m+1}^{N}(d_k^+(\zeta))^{-2} & (\delta_-(\zeta))^{-1}\delta_+(\zeta) \end{pmatrix},$$

and

$$\widehat{m}^{\sharp}_{+}(\zeta)=\begin{cases} \widehat{m}^{\sharp}_{-}(\zeta)\left(\mathrm{I}+\frac{C_n^{\mathcal{K}}}{(\zeta-\varsigma_n)}\sigma_+\right), & \zeta\in\widehat{\mathcal{K}}_n,\quad n\in\{m+1,m+2,\dots,N\}, \\ \widehat{m}^{\sharp}_{-}(\zeta)\left(\mathrm{I}+\frac{C_n^{\mathcal{L}}}{(\zeta-\overline{\varsigma_n})}\sigma_-\right), & \zeta\in\widehat{\mathcal{L}}_n,\quad n\in\{m+1,m+2,\dots,N\}; \end{cases}$$

(iii) $\widehat{m}^{\sharp}(\zeta)$ *has simple poles in* σ'_d *with*

$$\begin{aligned} \mathrm{Res}(\widehat{m}^{\sharp}(\zeta);\varsigma_n) &= \lim_{\zeta\to\varsigma_n}\widehat{m}^{\sharp}(\zeta)g_n(\delta(\varsigma_n))^{-2}\left(\prod_{k=m+1}^{N}(d_k^+(\varsigma_n))^{-2}\right)\sigma_-, && n\in\{1,2,\dots,m\}, \\ \mathrm{Res}(\widehat{m}^{\sharp}(\zeta);\overline{\varsigma_n}) &= \sigma_1\overline{\mathrm{Res}(\widehat{m}^{\sharp}(\zeta);\varsigma_n)}\,\sigma_1, && n\in\{1,2,\dots,m\}; \end{aligned}$$

(iv) $\det(\widehat{m}^{\sharp}(\zeta))|_{\zeta=\pm1}=0$;

(v) $\widehat{m}^{\sharp}(\zeta)=_{\zeta\to 0}\zeta^{-1}(\delta(0))^{\sigma_3}\left(\prod_{k=m+1}^{N}(d_k^+(0))^{\sigma_3}\right)\sigma_2+\mathcal{O}(1)$;

(vi) $\widehat{m}^{\sharp}(\zeta)\underset{\substack{\zeta\to\infty\\ \zeta\in\mathbb{C}\setminus\sigma'_{\mathcal{O}\mathcal{D}}}}{=}\mathrm{I}+\mathcal{O}(\zeta^{-1})$;

(vii) $\widehat{m}^{\sharp}(\zeta)=\sigma_1\overline{\widehat{m}^{\sharp}(\overline{\zeta})}\,\sigma_1$ *and* $\widehat{m}^{\sharp}(\zeta^{-1})=\zeta\widehat{m}^{\sharp}(\zeta)(\delta(0))^{\sigma_3}\left(\prod_{k=m+1}^{N}(d_k^+(0))^{\sigma_3}\right)\sigma_2$.

Let

$$u(x,t):=\mathrm{i}\lim_{\substack{\zeta\to\infty\\ \zeta\in\mathbb{C}\setminus\sigma'_{\mathcal{O}\mathcal{D}}}}\left(\zeta\left(\widehat{m}^{\sharp}(\zeta)(\delta(\zeta))^{\sigma_3}\prod_{k=m+1}^{N}(d_k^+(\zeta))^{\sigma_3}-\mathrm{I}\right)\right)_{12}, \tag{61}$$

and

$$\int_{+\infty}^{x}(|u(x',t)|^2-1)\,\mathrm{d}x':=-\mathrm{i}\lim_{\substack{\zeta\to\infty\\ \zeta\in\mathbb{C}\setminus\sigma'_{\mathcal{O}\mathcal{D}}}}\left(\zeta\left(\widehat{m}^{\sharp}(\zeta)(\delta(\zeta))^{\sigma_3}\prod_{k=m+1}^{N}(d_k^+(\zeta))^{\sigma_3}-\mathrm{I}\right)\right)_{11}. \tag{62}$$

Then $u(x,t)$ *is the solution of the Cauchy problem for the* D_fNLSE.

Proof. From the definition of $\widehat{m}^{\sharp}(\zeta)$ given in the Lemma, one shows that, for $m\in\{1,2,\dots,N\}$, $\widehat{m}^{\sharp}_{+}(\zeta)=\widehat{m}^{\sharp}_{-}(\zeta)\widehat{v}^{\sharp}_{\widehat{\mathcal{K}}_n}(\zeta)$, $\zeta\in\cup_{n=m+1}^{N}\widehat{\mathcal{K}}_n$, and $\widehat{m}^{\sharp}_{+}(\zeta)=\widehat{m}^{\sharp}_{-}(\zeta)\widehat{v}^{\sharp}_{\widehat{\mathcal{L}}_n}(\zeta)$, $\zeta\in\cup_{n=m+1}^{N}\widehat{\mathcal{L}}_n$, where

$$\widehat{v}^{\sharp}_{\widehat{\mathcal{K}}_n}(\zeta)=\left(\prod_{k=m+1}^{N}(d_k^{-}(\zeta))^{\sigma_3}\right)J_{\widehat{\mathcal{K}}_n}(\zeta)\left(\mathrm{I}+\tfrac{g_n(\delta(\varsigma_n))^{-2}}{(\zeta-\varsigma_n)}\sigma_{-}\right)\prod_{k=m+1}^{N}(d_k^{+}(\zeta))^{-\sigma_3},$$
$$\widehat{v}^{\sharp}_{\widehat{\mathcal{L}}_n}(\zeta)=\left(\prod_{k=m+1}^{N}(d_k^{+}(\zeta))^{\sigma_3}\right)\left(\mathrm{I}+\tfrac{\overline{g_n(\delta(\varsigma_n))^{-2}}}{(\zeta-\overline{\varsigma_n})}\sigma_{+}\right)(J_{\widehat{\mathcal{L}}_n}(\zeta))^{-1}\prod_{k=m+1}^{N}(d_k^{-}(\zeta))^{-\sigma_3}.$$

Now, as in [34], demanding that $\widehat{v}^{\sharp}_{\widehat{\mathcal{K}}_n}(\zeta)$ (respectively, $\widehat{v}^{\sharp}_{\widehat{\mathcal{L}}_n}(\zeta)$) have the following upper (respectively, lower) diagonal structure, $\widehat{v}^{\sharp}_{\widehat{\mathcal{K}}_n}(\zeta)=\mathrm{I}+C_n^{\mathcal{K}}(\zeta-\varsigma_n)^{-1}\sigma_{+}$ (respectively, $\widehat{v}^{\sharp}_{\widehat{\mathcal{L}}_n}(\zeta)=\mathrm{I}+C_n^{\mathcal{L}}(\zeta-\overline{\varsigma_n})^{-1}\sigma_{-}$), one arrives at

$$J_{\widehat{\mathcal{K}}_n}(\zeta)=\begin{pmatrix}\prod_{k=m+1}^{N}\frac{d_k^{+}(\zeta)}{d_k^{-}(\zeta)}-\frac{C_n^{\mathcal{K}}g_n(\delta(\varsigma_n))^{-2}}{(\zeta-\varsigma_n)^2}\prod_{k=m+1}^{N}\frac{(d_k^{+}(\zeta))^{-1}}{d_k^{-}(\zeta)} & \frac{C_n^{\mathcal{K}}}{(\zeta-\varsigma_n)}\prod_{k=m+1}^{N}\frac{(d_k^{+}(\zeta))^{-1}}{d_k^{-}(\zeta)}\\ -\frac{g_n(\delta(\varsigma_n))^{-2}}{(\zeta-\varsigma_n)}\prod_{k=m+1}^{N}\frac{d_k^{-}(\zeta)}{d_k^{+}(\zeta)} & \prod_{k=m+1}^{N}\frac{d_k^{-}(\zeta)}{d_k^{+}(\zeta)}\end{pmatrix},$$
$$J_{\widehat{\mathcal{L}}_n}(\zeta)=\begin{pmatrix}\prod_{k=m+1}^{N}\frac{d_k^{+}(\zeta)}{d_k^{-}(\zeta)} & \frac{\overline{g_n(\delta(\varsigma_n))^{-2}}}{(\zeta-\overline{\varsigma_n})}\prod_{k=m+1}^{N}\frac{d_k^{+}(\zeta)}{d_k^{-}(\zeta)}\\ -\frac{C_n^{\mathcal{L}}}{(\zeta-\overline{\varsigma_n})}\prod_{k=m+1}^{N}\frac{d_k^{-}(\zeta)}{(d_k^{+}(\zeta))^{-1}} & \prod_{k=m+1}^{N}\frac{d_k^{-}(\zeta)}{d_k^{+}(\zeta)}-\frac{C_n^{\mathcal{L}}\overline{g_n(\delta(\varsigma_n))^{-2}}}{(\zeta-\overline{\varsigma_n})^2}\prod_{k=m+1}^{N}\frac{d_k^{-}(\zeta)}{(d_k^{+}(\zeta))^{-1}}\end{pmatrix},$$

with $\det(J_{\widehat{\mathcal{K}}_n}(\zeta))=\det(J_{\widehat{\mathcal{L}}_n}(\zeta))=1$. Choosing $d_n^{\pm}(\zeta)$ as in the Lemma, one shows that

$$\mathrm{Res}(J_{\widehat{\mathcal{K}}_n}(\zeta);\varsigma_n)=\begin{pmatrix}\left.\left(\prod_{\substack{k=m+1\\k\neq n}}^{N}\frac{d_k^{+}(\zeta)}{d_k^{-}(\zeta)}-\frac{C_n^{\mathcal{K}}g_n(\delta(\varsigma_n))^{-2}}{(\zeta-\overline{\varsigma_n})^2}\prod_{\substack{k=m+1\\k\neq n}}^{N}\frac{(d_k^{+}(\zeta))^{-1}}{d_k^{-}(\zeta)}\right)\right|_{\zeta=\varsigma_n} & 0\\ 0 & 0\end{pmatrix},$$
$$\mathrm{Res}(J_{\widehat{\mathcal{L}}_n}(\zeta);\overline{\varsigma_n})=\begin{pmatrix}0 & 0\\ 0 & \left.\left(\prod_{\substack{k=m+1\\k\neq n}}^{N}\frac{d_k^{-}(\zeta)}{d_k^{+}(\zeta)}-\frac{C_n^{\mathcal{L}}\overline{g_n(\delta(\varsigma_n))^{-2}}}{(\zeta-\varsigma_n)^2}\prod_{\substack{k=m+1\\k\neq n}}^{N}\frac{d_k^{-}(\zeta)}{(d_k^{+}(\zeta))^{-1}}\right)\right|_{\zeta=\overline{\varsigma_n}}\end{pmatrix}:$$

choosing $C_n^{\mathcal{K}}$ and $C_n^{\mathcal{L}}$ as in the Lemma, one gets that $\mathrm{Res}(J_{\widehat{\mathcal{K}}_n}(\zeta);\varsigma_n)=\mathrm{Res}(J_{\widehat{\mathcal{L}}_n}(\zeta);\overline{\varsigma_n})=\mathbf{0}$; thus, $J_{\widehat{\mathcal{K}}_n}(\zeta)$ (respectively, $J_{\widehat{\mathcal{L}}_n}(\zeta)$) is holomorphic in $\cup_{n=m+1}^{N}\mathrm{int}(\widehat{\mathcal{K}}_n)$ (respectively, $\cup_{n=m+1}^{N}\mathrm{int}(\widehat{\mathcal{L}}_n)$). The remainder of the proof follows from Lemma 3.2 and the definition of $\widehat{m}^{\sharp}(\zeta)$ given in the Lemma via straightforward algebraic calculations. □

Remark 3.4. One notes from the proof of Lemma 3.3 that, for $m\in\{1,2,\dots,N\}$, with $\eta_n:=\sin(\phi_n)\in(0,1)$ and $\xi_n:=\cos(\phi_n)\in(-1,1)$, as $t\to+\infty$ and $x\to-\infty$ such that $z_o:=x/t<-2$ and $(x,t)\in\urcorner_m$,

$$\widehat{v}^{\sharp}_{\widehat{\mathcal{K}}_n}(\zeta)=\mathrm{I}+\tfrac{C_n^{\mathcal{K}}}{(\zeta-\varsigma_n)}\sigma_{+}=\mathrm{I}+\mathcal{O}\left(\tfrac{\mathrm{e}^{-4t\eta_n|\xi_n-\xi_m|}}{(\zeta-\varsigma_n)}\sigma_{+}\right),\quad \zeta\in\widehat{\mathcal{K}}_n,\quad n\in\{m+1,m+2,\dots,N\},$$
$$\widehat{v}^{\sharp}_{\widehat{\mathcal{L}}_n}(\zeta)=\mathrm{I}+\tfrac{C_n^{\mathcal{L}}}{(\zeta-\overline{\varsigma_n})}\sigma_{-}=\mathrm{I}+\mathcal{O}\left(\tfrac{\mathrm{e}^{-4t\eta_n|\xi_n-\xi_m|}}{(\zeta-\overline{\varsigma_n})}\sigma_{-}\right),\quad \zeta\in\widehat{\mathcal{L}}_n,\quad n\in\{m+1,m+2,\dots,N\};$$

hence, as $t\to+\infty$, $\widehat{v}^{\sharp}_{\widehat{\star}_n}(\zeta)\to\mathrm{I}$ (uniformly), where $\star\in\{\mathcal{K},\mathcal{L}\}$. One also notes from Lemmae 3.1–3.3 that, for $\zeta\in\cup_{n=m+1}^{N}\mathrm{int}(\widehat{\mathcal{K}}_n)$,

$$\widehat{m}^{\sharp}(\zeta)=\widehat{m}(\zeta)\begin{pmatrix}\left(\frac{\zeta-\varsigma_n}{\zeta-\overline{\varsigma_n}}\right)\prod_{\substack{k=m+1\\k\neq n}}^{N}(d_k^{+}(\zeta))^{-1} & -\frac{C_n^{\mathcal{K}}}{(\zeta-\overline{\varsigma_n})}\prod_{\substack{k=m+1\\k\neq n}}^{N}(d_k^{+}(\zeta))^{-1}\\ 0 & \left(\frac{\zeta-\overline{\varsigma_n}}{\zeta-\varsigma_n}\right)\prod_{\substack{k=m+1\\k\neq n}}^{N}d_k^{+}(\zeta)\end{pmatrix},$$

and, for $\zeta\in\cup_{n=m+1}^{N}\mathrm{int}(\widehat{\mathcal{L}}_n)$,

$$\widehat{m}^{\sharp}(\zeta)=\widehat{m}(\zeta)\begin{pmatrix}\left(\frac{\zeta-\varsigma_n}{\zeta-\overline{\varsigma_n}}\right)\prod_{\substack{k=m+1\\k\neq n}}^{N}(d_k^{+}(\zeta))^{-1} & 0\\ \frac{C_n^{\mathcal{L}}}{(\zeta-\varsigma_n)}\prod_{\substack{k=m+1\\k\neq n}}^{N}d_k^{+}(\zeta) & \left(\frac{\zeta-\overline{\varsigma_n}}{\zeta-\varsigma_n}\right)\prod_{\substack{k=m+1\\k\neq n}}^{N}d_k^{+}(\zeta)\end{pmatrix};$$

hence, modulo singular terms like $(\zeta-\varsigma_n)^{-1}$ and $(\zeta-\overline{\varsigma_n})^{-1}$, and recalling that (see above), as $t\to+\infty$, $C_n^{\mathcal{K}}$ and $C_n^{\mathcal{L}}$ are $\mathcal{O}(\exp(-4t\eta_n|\xi_n-\xi_m|))$, one deduces that, since the RHP for $\widehat{m}(\zeta)$ formulated in Lemma 3.1 is asymptotically solvable [38], there are no exponentially growing factors for $\widehat{m}^{\sharp}(\zeta)$ when $\zeta\in\cup_{n=m+1}^{N}(\mathrm{int}(\widehat{\mathcal{K}}_n)\cup\mathrm{int}(\widehat{\mathcal{L}}_n))$.

By estimating the error along the trajectory of the mth dark soliton ($m \in \{1,2,\dots,N\}$) when the jump matrices on $\{\widehat{\mathcal{K}}_n, \widehat{\mathcal{L}}_n\}_{n=m+1}^{N}$ are removed from the specification of the RHP for $\widehat{m}^{\sharp}(\zeta)$, one arrives at an asymptotically solvable, model RHP (see Lemma 3.5 below); however, since the proof of Lemma 3.5 relies substantially on the Beals-Coifman (BC) construction [41] for the solution of a matrix (and appropriately normalised) RHP on an oriented and unbounded contour, it is convenient to present, with some requisite preamble, a succinct and self-contained synopsis of it at this juncture. But first, the following result is necessary.

Proposition 3.1 ([38])**.** *The solution of the* RHP *for* $\widehat{m}^{\sharp}(\zeta)\colon \mathbb{C} \setminus \sigma'_{\mathcal{O}\mathcal{D}} \to \mathrm{M}_2(\mathbb{C})$ *formulated in Lemma* 3.3 *has the (integral equation) representation*

$$\widehat{m}^{\sharp}(\zeta) = \left(\mathrm{I} + \zeta^{-1}\widehat{\Delta}_o^{\sharp}\right)\widehat{\mathcal{P}}^{\sharp}(\zeta)\left(\widehat{m}_d^{\sharp}(\zeta) + \int_{\sigma'_c} \frac{\widehat{m}_-^{\sharp}(\mu)(\widehat{\upsilon}^{\sharp}(\mu)-\mathrm{I})}{(\mu-\zeta)}\frac{\mathrm{d}\mu}{2\pi\mathrm{i}}\right), \quad \zeta \in \mathbb{C} \setminus \sigma'_{\mathcal{O}\mathcal{D}},$$

where

$$\widehat{m}_d^{\sharp}(\zeta) = \mathrm{I} + \sum_{n=1}^{m}\left(\frac{\mathrm{Res}(\widehat{m}^{\sharp}(\zeta);\varsigma_n)}{(\zeta-\varsigma_n)} + \frac{\sigma_1\overline{\mathrm{Res}(\widehat{m}^{\sharp}(\zeta);\varsigma_n)}\,\sigma_1}{(\zeta-\overline{\varsigma_n})}\right),$$

$\widehat{\upsilon}^{\sharp}(\cdot)$ *is a generic notation for the jump matrices of* $\widehat{m}^{\sharp}(\zeta)$ *on* σ'_c *(Lemma* 3.3*, (ii)), and* $\widehat{\Delta}_o^{\sharp}$ *and* $\widehat{\mathcal{P}}^{\sharp}(\zeta)$ *are specified below. The solution of the above (integral) equation can be written as the ordered factorisation*

$$\widehat{m}^{\sharp}(\zeta) = \left(\mathrm{I} + \zeta^{-1}\widehat{\Delta}_o^{\sharp}\right)\widehat{\mathcal{P}}^{\sharp}(\zeta)\widehat{m}_d^{\sharp}(\zeta)m^c(\zeta), \quad \zeta \in \mathbb{C} \setminus \sigma'_{\mathcal{O}\mathcal{D}},$$

where $\widehat{m}_d^{\sharp}(\zeta) = \sigma_1\overline{\widehat{m}_d^{\sharp}(\overline{\zeta})}\,\sigma_1$ ($\in \mathrm{SL}(2,\mathbb{C})$) *has the representation given above,* $\widehat{\mathcal{P}}^{\sharp}(\zeta) = \sigma_1\overline{\widehat{\mathcal{P}}^{\sharp}(\overline{\zeta})}\,\sigma_1$ *is chosen so that* $\widehat{\Delta}_o^{\sharp}$ *is idempotent,* $\mathrm{I} + \zeta^{-1}\widehat{\Delta}_o^{\sharp}$ ($\in \mathrm{M}_2(\mathbb{C})$) *is holomorphic in a punctured neighbourhood of the origin, with* $\widehat{\Delta}_o^{\sharp} = \sigma_1\overline{\widehat{\Delta}_o^{\sharp}}\,\sigma_1$ ($\in \mathrm{GL}(2,\mathbb{C})$) *such that* $\det(\mathrm{I}+\zeta^{-1}\widehat{\Delta}_o^{\sharp})|_{\zeta=\pm 1} = 0$*, and having the finite, order 2, matrix involutive structure*

$$\widehat{\Delta}_o^{\sharp} = \begin{pmatrix} \widehat{\Delta}^{\sharp}\mathrm{e}^{\mathrm{i}(k+1/2)\pi} & (1+(\widehat{\Delta}^{\sharp})^2)^{1/2}\mathrm{e}^{-\mathrm{i}\widehat{\vartheta}^{\sharp}} \\ (1+(\widehat{\Delta}^{\sharp})^2)^{1/2}\mathrm{e}^{\mathrm{i}\widehat{\vartheta}^{\sharp}} & \widehat{\Delta}^{\sharp}\mathrm{e}^{-\mathrm{i}(k+1/2)\pi} \end{pmatrix}, \quad k \in \mathbb{Z},$$

where $\widehat{\Delta}^{\sharp}$ *and* $\widehat{\vartheta}^{\sharp}$ *are obtained from the relation* $\widehat{\Delta}_o^{\sharp} = \widehat{\mathcal{P}}^{\sharp}(0)\widehat{m}_d^{\sharp}(0)m^c(0)(\delta(0))^{\sigma_3}\left(\prod_{k=m+1}^{N}(d_k^+(0))^{\sigma_3}\right)\sigma_2$*, and satisfying* $\mathrm{tr}(\widehat{\Delta}_o^{\sharp}) = 0$*,* $\det(\widehat{\Delta}_o^{\sharp}) = -1$*, and* $\widehat{\Delta}_o^{\sharp}\widehat{\Delta}_o^{\sharp} = \mathrm{I}$*, and* $m^c(\zeta)\colon \mathbb{C} \setminus \sigma'_c \to \mathrm{SL}(2,\mathbb{C})$ *solves the following* RHP*: (1)* $m^c(\zeta)$ *is piecewise (sectionally) holomorphic* $\forall\, \zeta \in \mathbb{C} \setminus \sigma'_c$*; (2)* $m_{\pm}^c(\zeta) := \lim_{\substack{\zeta' \to \zeta \\ \zeta' \in \pm \text{ side of } \sigma'_c}} m^c(\zeta')$ *satisfy, for* $\zeta \in \sigma'_c$*, the jump condition* $m_+^c(\zeta) = m_-^c(\zeta)\upsilon^c(\zeta)$*, where* $\upsilon^c(\zeta) = \exp(-\mathrm{i}k(\zeta)(x+2\lambda(\zeta)t)\mathrm{ad}(\sigma_3))\widehat{\mathcal{G}}^{\sharp}(\zeta)$*,* $\zeta \in \mathbb{R}$*, with* $\widehat{\mathcal{G}}^{\sharp}(\zeta)$ *given in Lemma* 3.3*, (ii),* $\upsilon^c(\zeta) = \mathrm{I} + C_n^{\mathcal{K}}(\zeta - \varsigma_n)^{-1}\sigma_+$*,* $\zeta \in \widehat{\mathcal{K}}_n$*, and* $\upsilon^c(\zeta) = \mathrm{I} + C_n^{\mathcal{L}}(\zeta - \overline{\varsigma_n})^{-1}\sigma_-$*,* $\zeta \in \widehat{\mathcal{L}}_n$*,* $n \in \{m+1, m+2, \dots, N\}$*, with* $C_n^{\mathcal{K}}$ *and* $C_n^{\mathcal{L}}$ *given in Lemma* 3.3*; (3)* $m^c(\zeta) \underset{\substack{\zeta\to\infty \\ \zeta\in\mathbb{C}\setminus\sigma'_c}}{=} \mathrm{I} + \mathcal{O}(\zeta^{-1})$*; and (4)* $m^c(\zeta) = \sigma_1\overline{m^c(\overline{\zeta})}\,\sigma_1$*.*

The BC formulation [41] now follows. One agrees to call a contour $\Gamma^{\sharp}$ oriented if: (1) $\mathbb{C} \setminus \Gamma^{\sharp}$ has finitely many open connected components; (2) $\mathbb{C}\setminus\Gamma^{\sharp}$ is the disjoint union of two, possibly disconnected, open regions, denoted by $\mho^+$ and $\mho^-$; and (3) $\Gamma^{\sharp}$ may be viewed as either the positively oriented boundary for $\mho^+$ or the negatively oriented boundary for $\mho^-$ ($\mathbb{C} \setminus \Gamma^{\sharp}$ is coloured by two colours, $\pm$). Let $\Gamma^{\sharp}$, as a closed set, be the union of finitely many oriented simple piecewise-smooth arcs. Denote the set of all self-intersections of $\Gamma^{\sharp}$ by $\widehat{\Gamma}^{\sharp}$ (with $\mathrm{card}(\widehat{\Gamma}^{\sharp}) < \infty$ assumed throughout). Set $\widetilde{\Gamma}^{\sharp} := \Gamma^{\sharp} \setminus \widehat{\Gamma}^{\sharp}$. The BC construction for the solution of a (matrix) RHP, in the absence of a discrete spectrum and spectral singularities [45, 53], on an oriented contour $\Gamma^{\sharp}$ consists of finding an $\mathrm{M}_2(\mathbb{C})$-valued function $\mathcal{X}(\lambda)$ such that: (1) $\mathcal{X}(\lambda)$ is piecewise holomorphic $\forall\,\lambda \in \mathbb{C}\setminus\Gamma^{\sharp}$; (2) $\mathcal{X}_+(\lambda) = \mathcal{X}_-(\lambda)\upsilon(\lambda)$, $\lambda \in \widetilde{\Gamma}^{\sharp}$, for some "jump" matrix $\upsilon(\lambda)\colon \widetilde{\Gamma}^{\sharp} \to \mathrm{GL}(2,\mathbb{C})$; and (3) uniformly as $\lambda \to \infty$, $\lambda \in \mathbb{C} \setminus \Gamma^{\sharp}$, $\mathcal{X}(\lambda) = \mathrm{I} + \mathcal{O}(\lambda^{-1})$. Let $\upsilon(\lambda) := (\mathrm{I} - w_-(\lambda))^{-1}(\mathrm{I} + w_+(\lambda))$, $\lambda \in \widetilde{\Gamma}^{\sharp}$, be a factorisation for $\upsilon(\lambda)$, where $w_{\pm}(\lambda)$ are some upper/lower, or lower/upper, triangular (depending on the orientation of $\Gamma^{\sharp}$) nilpotent matrices, with degree of nilpotency 2, and $w_{\pm}(\lambda) \in \cap_{p\in\{2,\infty\}}\mathcal{L}^p_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Gamma}^{\sharp})$ (if $\widetilde{\Gamma}^{\sharp}$ is unbounded, one requires that $w_{\pm}(\lambda) \underset{\substack{\lambda\to\infty \\ \lambda\in\widetilde{\Gamma}^{\sharp}}}{=} \mathbf{0}$). Define $w(\lambda) := w_+(\lambda) + w_-(\lambda)$, and introduce the Cauchy operators on $\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Gamma^{\sharp})$, $(C_{\pm}f)(\lambda) :=$

$\lim_{\substack{\lambda' \to \lambda \\ \lambda' \in \pm \text{ side of } \Gamma^\sharp}} \int_{\Gamma^\sharp} \frac{f(z)}{(z-\lambda')} \frac{\mathrm{d}z}{2\pi\mathrm{i}}$, where $f(\cdot) \in \mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Gamma^\sharp)$, with $C_\pm \colon \mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Gamma^\sharp) \to \mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Gamma^\sharp)$ bounded in operator norm[1], and $||(C_\pm f)(\cdot)||_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(*)} \leqslant \text{const.} ||f(\cdot)||_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(*)}$. Introduce the BC operator:

$$C_w f := C_+(f w_-) + C_-(f w_+), \qquad f(\cdot) \in \mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(*);$$

moreover, since $\mathbb{C} \setminus \Gamma^\sharp$ can be coloured by two colours ($\pm$), $C_\pm$ are complementary projections [45], namely, $C_+^2 = C_+$, $C_-^2 = -C_-$, $C_+ C_- = C_- C_+ = \underline{\mathbf{0}}$ (the null operator), and $C_+ - C_- = \mathbf{id}$ (the identity operator): in the case that C_+ and $-C_-$ are complementary, the contour $\Gamma^\sharp$ can always be oriented in such a way that the $\pm$ regions lie on the $\pm$ sides of the contour, respectively. Specialising the BC construction to the solution of the RHP for $m^c(\zeta)$ on σ_c' formulated in Proposition 3.1, and writing $\upsilon^c(\zeta)$ as the following (bounded) algebraic factorisation $\upsilon^c(\zeta) := (\mathrm{I} - w_-^c(\zeta))^{-1}(\mathrm{I} + w_+^c(\zeta))$, $\zeta \in \sigma_c'$, the integral representation for $m^c(\zeta)$ is given by the following

Lemma 3.4 (Beals and Coifman [41]**).** *Let*

$$\mu^c(\zeta) = m_+^c(\zeta)(\mathrm{I} + w_+^c(\zeta))^{-1} = m_-^c(\zeta)(\mathrm{I} - w_-^c(\zeta))^{-1}, \quad \zeta \in \sigma_c'.$$

If $\mu^c(\zeta) \in \mathrm{I} + \mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\sigma_c') := \{\mathrm{I} + h(\cdot);\, h(\cdot) \in \mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\sigma_c')\}$[2] *solves the linear singular integral equation*

$$(\mathbf{id} - C_{w^c})(\mu^c(\zeta) - \mathrm{I}) = C_{w^c}\mathrm{I} = C_+(w_-^c) + C_-(w_+^c), \quad \zeta \in \sigma_c',$$

where $\mathbf{id}$ *is the identity operator on* $\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\sigma_c')$*, then the solution of the* RHP *for* $m^c(\zeta)$ *is*

$$m^c(\zeta) = \mathrm{I} + \int_{\sigma_c'} \frac{\mu^c(z) w^c(z)}{(z-\zeta)} \frac{\mathrm{d}z}{2\pi\mathrm{i}}, \qquad \zeta \in \mathbb{C} \setminus \sigma_c',$$

where $\mu^c(\zeta) = ((\mathbf{id} - C_{w^c})^{-1}\mathrm{I})(\zeta)$*, and* $w^c(\zeta) := w_+^c(\zeta) + w_-^c(\zeta)$.

Finally, one arrives at, and is in a position to prove, the following

Lemma 3.5. *For* $m \in \{1, 2, \dots, N\}$*, set* $\widehat{\sigma}_d := \cup_{n=1}^m (\{\varsigma_n\} \cup \{\overline{\varsigma_n}\})$*, and let* $\sigma_c = \{\zeta;\, \mathrm{Im}(\zeta) = 0\}$ *with orientation from* $-\infty$ *to* $+\infty$*. Let* $\widehat{\chi}(\zeta) \colon \mathbb{C} \setminus (\widehat{\sigma}_d \cup \sigma_c) \to \mathrm{M}_2(\mathbb{C})$ *solve the following* RHP:

(i) $\widehat{\chi}(\zeta)$ *is piecewise (sectionally) meromorphic* $\forall\, \zeta \in \mathbb{C} \setminus \sigma_c$;

(ii) $\widehat{\chi}_\pm(\zeta) := \lim_{\substack{\zeta' \to \zeta \\ \zeta' \in \pm \text{ side of } \sigma_c}} \widehat{\chi}(\zeta')$ *satisfy the jump condition*

$$\widehat{\chi}_+(\zeta) = \widehat{\chi}_-(\zeta) \exp(-\mathrm{i}k(\zeta)(x + 2\lambda(\zeta)t)\mathrm{ad}(\sigma_3)) \widehat{\mathcal{G}}^\sharp(\zeta), \quad \zeta \in \mathbb{R};$$

(iii) $\widehat{\chi}(\zeta)$ *has simple poles in* $\widehat{\sigma}_d$ *with*

$$\begin{aligned} \mathrm{Res}(\widehat{\chi}(\zeta); \varsigma_n) &= \lim_{\zeta \to \varsigma_n} \widehat{\chi}(\zeta) g_n(\delta(\varsigma_n))^{-2} \left(\prod_{k=m+1}^N (d_k^+(\varsigma_n))^{-2} \right) \sigma_-, && n \in \{1, 2, \dots, m\}, \\ \mathrm{Res}(\widehat{\chi}(\zeta); \overline{\varsigma_n}) &= \sigma_1 \overline{\mathrm{Res}(\widehat{\chi}(\zeta); \varsigma_n)}\, \sigma_1, && n \in \{1, 2, \dots, m\}; \end{aligned}$$

(iv) $\det(\widehat{\chi}(\zeta))|_{\zeta = \pm 1} = 0$;

(v) $\widehat{\chi}(\zeta) =_{\zeta \to 0} \zeta^{-1}(\delta(0))^{\sigma_3} \left(\prod_{k=m+1}^N (d_k^+(0))^{\sigma_3} \right) \sigma_2 + \mathcal{O}(1)$;

(vi) $\widehat{\chi}(\zeta) \underset{\zeta \in \mathbb{C} \setminus (\widehat{\sigma}_d \cup \sigma_c)}{\overset{\zeta \to \infty}{=}} \mathrm{I} + \mathcal{O}(\zeta^{-1})$;

(vii) $\widehat{\chi}(\zeta) = \sigma_1 \overline{\widehat{\chi}(\overline{\zeta})}\, \sigma_1$ *and* $\widehat{\chi}(\zeta^{-1}) = \zeta \widehat{\chi}(\zeta)(\delta(0))^{\sigma_3} \left(\prod_{k=m+1}^N (d_k^+(0))^{\sigma_3} \right) \sigma_2$.

[1] $||C_\pm||_{\mathcal{N}(\Gamma^\sharp)} < \infty$, where $\mathcal{N}(*)$ denotes the space of all bounded linear operators acting from $\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(*)$ into $\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(*)$.

[2] For $f(\zeta) \in \mathrm{I} + \mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(*)$, $||f(\cdot)||_{\mathrm{I} + \mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(*)} := \left(||f(\infty)||^2_{\mathcal{L}^\infty_{\mathrm{M}_2(\mathbb{C})}(*)} + ||f(\cdot) - f(\infty)||^2_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(*)} \right)^{1/2}$ [44].

Then, as $t\to+\infty$ and $x\to-\infty$ such that $z_o:=x/t<-2$ and $(x,t)\in\daleth_m$, $\widehat{m}^{\sharp}(\zeta)\colon\mathbb{C}\setminus\sigma'_{\mathcal{O}^{\mathcal{D}}}\to\mathrm{M}_2(\mathbb{C})$ has the following asymptotics:

$$\widehat{m}^{\sharp}(\zeta)=\Big(\mathrm{I}+\mathcal{O}\Big(\widehat{\mathfrak{F}}(\zeta)\exp\Big(-\widehat{\beth}\,t\Big)\Big)\Big)\widehat{\chi}(\zeta),$$

where $\widehat{\beth}:=4\min_{\substack{m\in\{1,2,\dots,N\}\\ n\in\{m+1,m+2,\dots,N\}}}\{\sin(\phi_n)|\cos(\phi_n)-\cos(\phi_m)|\}$ (>0), and, for $i,j\in\{1,2\}$, $(\widehat{\mathfrak{F}}(\zeta))_{ij}=_{\zeta\to\infty}\mathcal{O}(|\zeta|^{-1})$ and $(\widehat{\mathfrak{F}}(\zeta))_{ij}=_{\zeta\to0}\mathcal{O}(1)$. Furthermore, let

$$u(x,t):=\mathrm{i}\lim_{\substack{\zeta\to\infty\\ \zeta\in\mathbb{C}\setminus(\widehat{\sigma}_d\cup\sigma_c)}}\left(\zeta\left(\widehat{\chi}(\zeta)(\delta(\zeta))^{\sigma_3}\prod_{k=m+1}^{N}(d_k^{+}(\zeta))^{\sigma_3}-\mathrm{I}\right)\right)_{12}+\mathcal{O}\Big(\exp\Big(-\widehat{\beth}\,t\Big)\Big),\tag{63}$$

and

$$\int_{+\infty}^{x}(|u(x',t)|^2-1)\,\mathrm{d}x':=-\mathrm{i}\lim_{\substack{\zeta\to\infty\\ \zeta\in\mathbb{C}\setminus(\widehat{\sigma}_d\cup\sigma_c)}}\left(\zeta\left(\widehat{\chi}(\zeta)(\delta(\zeta))^{\sigma_3}\prod_{k=m+1}^{N}(d_k^{+}(\zeta))^{\sigma_3}-\mathrm{I}\right)\right)_{11}+\mathcal{O}\Big(\exp\Big(-\widehat{\beth}\,t\Big)\Big).\tag{64}$$

Then $u(x,t)$ is the solution of the Cauchy problem for the D_fNLSE.

Remark 3.5. The solution of the (normalised at ∞) RHP for $\widehat{\chi}(\zeta)\colon\mathbb{C}\setminus(\widehat{\sigma}_d\cup\sigma_c)\to\mathrm{M}_2(\mathbb{C})$ formulated in Lemma 3.5 has a factorised representation analogous to that of $\widehat{m}^{\sharp}(\zeta)$ given in Proposition 3.1 (with appropriate change(s) of notation).

Proof. Define $\mathcal{E}(\zeta):=\widehat{m}^{\sharp}(\zeta)(\widehat{\chi}(\zeta))^{-1}$. From this definition, the RHPs for $\widehat{m}^{\sharp}(\zeta)$ and $\widehat{\chi}(\zeta)$ formulated in Lemmae 3.3 and 3.5, respectively, Proposition 3.1, and Remark 3.5, one shows that, for $m\in\{1,2,\dots,N\}$ and $n\in\{m+1,m+2,\dots,N\}$, $\mathcal{E}(\zeta)$ solves the following RHP: (1) $\mathcal{E}(\zeta)$ is piecewise (sectionally) holomorphic $\forall\,\zeta\in\mathbb{C}\setminus\Sigma_{\mathcal{E}}$, where $\Sigma_{\mathcal{E}}=\cup_{n=m+1}^{N}\Sigma_{\mathcal{E}}^{n}$, with $\Sigma_{\mathcal{E}}^{n}:=\widehat{\mathcal{K}}_n\cup\widehat{\mathcal{L}}_n$ (with orientations preserved); (2) $\mathcal{E}_{\pm}(\zeta):=\lim_{\substack{\zeta'\to\zeta\\ \zeta'\,\in\,\pm\,\text{side of}\,\Sigma_{\mathcal{E}}}}\mathcal{E}(\zeta')$ satisfy the jump condition $\mathcal{E}_{+}(\zeta)=\mathcal{E}_{-}(\zeta)\upsilon_{\mathcal{E}}(\zeta)$, $\zeta\in\Sigma_{\mathcal{E}}$, where

$$\upsilon_{\mathcal{E}}(\zeta)=\begin{cases}\mathrm{I}+\widetilde{\mathcal{W}}_{\mathcal{E}}^{\widehat{\mathcal{K}}_n}(\zeta), & \zeta\in\cup_{n=m+1}^{N}\widehat{\mathcal{K}}_n\,(\subset\Sigma_{\mathcal{E}}), \quad n\in\{m+1,m+2,\dots,N\},\\ \mathrm{I}+\widetilde{\mathcal{W}}_{\mathcal{E}}^{\widehat{\mathcal{L}}_n}(\zeta), & \zeta\in\cup_{n=m+1}^{N}\widehat{\mathcal{L}}_n\,(\subset\Sigma_{\mathcal{E}}), \quad n\in\{m+1,m+2,\dots,N\},\end{cases}$$

with $\widetilde{\mathcal{W}}_{\mathcal{E}}^{\widehat{\mathcal{K}}_n}(\zeta)=C_n^{\mathcal{K}}(\zeta-\varsigma_n)^{-1}\mathcal{X}^{\flat}(\zeta)$, $\widetilde{\mathcal{W}}_{\mathcal{E}}^{\widehat{\mathcal{L}}_n}(\zeta)=C_n^{\mathcal{L}}(\zeta-\overline{\varsigma_n})^{-1}\mathcal{X}^{\natural}(\zeta)$,

$$\mathcal{X}^{\flat}(\zeta)=\begin{pmatrix}-\widehat{\chi}_{11}(\zeta)\widehat{\chi}_{21}(\zeta) & (\widehat{\chi}_{11}(\zeta))^2\\ -(\widehat{\chi}_{21}(\zeta))^2 & \widehat{\chi}_{11}(\zeta)\widehat{\chi}_{21}(\zeta)\end{pmatrix},\qquad \mathcal{X}^{\natural}(\zeta)=\begin{pmatrix}\widehat{\chi}_{12}(\zeta)\widehat{\chi}_{22}(\zeta) & -(\widehat{\chi}_{12}(\zeta))^2\\ (\widehat{\chi}_{22}(\zeta))^2 & -\widehat{\chi}_{12}(\zeta)\widehat{\chi}_{22}(\zeta)\end{pmatrix},$$

and $C_n^{\mathcal{K}}$ and $C_n^{\mathcal{L}}$ given in Lemma 3.3; (3) $\det(\mathcal{E}(\zeta))|_{\zeta=\pm1}=1$; (4) $\mathcal{E}(\zeta)=_{\zeta\to0}\mathcal{O}(1)$ and $\mathcal{E}=_{\substack{\zeta\to\infty\\ \zeta\in\mathbb{C}\setminus\Sigma_{\mathcal{E}}}}\mathrm{I}+\mathcal{O}(\zeta^{-1})$; and (5) $\mathcal{E}(\zeta)=\sigma_1\overline{\mathcal{E}(\overline{\zeta})}\,\sigma_1$ and $\mathcal{E}(\zeta^{-1})=\mathcal{E}(\zeta)$. Note, in particular, that $\mathcal{E}(\zeta)$ has no jump discontinuity on $\mathbb{R}$, and no poles. Recall, now, the BC construction (see the paragraph preceding Lemma 3.4). Write the following (bounded) algebraic factorisation for $\upsilon_{\mathcal{E}}(\zeta)$, $\upsilon_{\mathcal{E}}(\zeta)=(\mathrm{I}-w_{-}^{\mathcal{E}}(\zeta))^{-1}(\mathrm{I}+w_{+}^{\mathcal{E}}(\zeta))$, $\zeta\in\Sigma_{\mathcal{E}}$, and choose [46] $w_{-}^{\mathcal{E}}(\zeta)=\mathbf{0}$; hence, $w_{+}^{\mathcal{E}}(\zeta)=\widetilde{\mathcal{W}}_{\mathcal{E}}^{\widehat{\mathcal{K}}_n}(\zeta)$, $\zeta\in\cup_{n=m+1}^{N}\widehat{\mathcal{K}}_n$, and $w_{+}^{\mathcal{E}}(\zeta)=\widetilde{\mathcal{W}}_{\mathcal{E}}^{\widehat{\mathcal{L}}_n}(\zeta)$, $\zeta\in\cup_{n=m+1}^{N}\widehat{\mathcal{L}}_n$. Let $\mu^{\mathcal{E}}(\zeta)$ be the solution of the BC linear singular integral equation $(\mathbf{id}_{\mathcal{E}}-C_{w^{\mathcal{E}}})\mu^{\mathcal{E}}(\zeta)=\mathrm{I}$, $\zeta\in\Sigma_{\mathcal{E}}$, where $\mathbf{id}_{\mathcal{E}}$ is the identity operator on $\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma_{\mathcal{E}})$, and, for $f(\cdot)\in\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\Sigma_{\mathcal{E}})$, set $C_{w^{\mathcal{E}}}f:=C_{+}(fw_{-}^{\mathcal{E}})+C_{-}(fw_{+}^{\mathcal{E}})=C_{-}(fw_{+}^{\mathcal{E}})$, with $(C_{\pm}f)(\zeta):=\lim_{\substack{\zeta'\to\zeta\\ \zeta'\,\in\,\pm\,\text{side of}\,\Sigma_{\mathcal{E}}}}\int_{\Sigma_{\mathcal{E}}}\frac{f(z)}{(z-\zeta')}\frac{\mathrm{d}z}{2\pi\mathrm{i}}$. It was shown in [38] that $\|(\mathbf{id}_{\mathcal{E}}-C_{w^{\mathcal{E}}})^{-1}\|_{\mathcal{N}(\Sigma_{\mathcal{E}})}<\infty$, where $\mathcal{N}(*)$ denotes the space of bounded linear operators from $\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(*)$ to $\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(*)$. According to the BC construction, the solution of the (normalised at ∞) RHP for $\mathcal{E}(\zeta)$ has the integral representation $\mathcal{E}(\zeta)=\mathrm{I}+\int_{\Sigma_{\mathcal{E}}}\frac{\mu^{\mathcal{E}}(z)w^{\mathcal{E}}(z)}{(z-\zeta)}\frac{\mathrm{d}z}{2\pi\mathrm{i}}$, $\zeta\in\mathbb{C}\setminus\Sigma_{\mathcal{E}}$, where $\mu^{\mathcal{E}}(\zeta)=((\mathbf{id}_{\mathcal{E}}-C_{w^{\mathcal{E}}})^{-1}\mathrm{I})(\zeta)$, and $w^{\mathcal{E}}(\zeta)=\sum_{l\in\{\pm\}}w_l^{\mathcal{E}}(\zeta)=w_{+}^{\mathcal{E}}(\zeta)$. Since (cf. Definition 3.1), for $i\neq j\in\{m+1,m+2,\dots,N\}$, $\widehat{\mathcal{K}}_i\cap\widehat{\mathcal{L}}_i=\widehat{\mathcal{K}}_i\cap\widehat{\mathcal{K}}_j=\widehat{\mathcal{L}}_i\cap\widehat{\mathcal{L}}_j=\emptyset$, it follows that

$$\mathcal{E}(\zeta)=\mathrm{I}+\sum_{n=m+1}^{N}\left(\int_{\widehat{\mathcal{K}}_n}\frac{\mu^{\mathcal{E}}(z)\widetilde{\mathcal{W}}_{\mathcal{E}}^{\widehat{\mathcal{K}}_n}(z)}{(z-\zeta)}\frac{\mathrm{d}z}{2\pi\mathrm{i}}+\int_{\widehat{\mathcal{L}}_n}\frac{\mu^{\mathcal{E}}(z)\widetilde{\mathcal{W}}_{\mathcal{E}}^{\widehat{\mathcal{L}}_n}(z)}{(z-\zeta)}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\right),\quad \zeta\in\mathbb{C}\setminus\Sigma_{\mathcal{E}}.$$

From the second resolvent identity and the expressions for $\widetilde{\mathcal{W}}_{\mathcal{E}}^{\widehat{\mathcal{K}}_n}(\zeta)$ and $\widetilde{\mathcal{W}}_{\mathcal{E}}^{\widehat{\mathcal{L}}_n}(\zeta)$, one shows that

$$\mathcal{E}(\zeta)-\mathrm{I}=\sum_{n=m+1}^{N}\left(\int_{\widehat{\mathcal{K}}_n}\frac{C_n^{\mathcal{K}}\mathcal{X}^{\flat}(z)}{(z-\varsigma_n)(z-\zeta)}\frac{\mathrm{d}z}{2\pi\mathrm{i}}+\int_{\widehat{\mathcal{L}}_n}\frac{C_n^{\mathcal{L}}\mathcal{X}^{\natural}(z)}{(z-\overline{\varsigma_n})(z-\zeta)}\frac{\mathrm{d}z}{2\pi\mathrm{i}}+\int_{\widehat{\mathcal{K}}_n}\frac{C_n^{\mathcal{K}}((\mathbf{id}_{\mathcal{E}}-C_{w^{\mathcal{E}}})^{-1}C_{w^{\mathcal{E}}}\mathrm{I})(z)\mathcal{X}^{\flat}(z)}{(z-\varsigma_n)(z-\zeta)}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\right.$$
$$\left.+\int_{\widehat{\mathcal{L}}_n}\frac{C_n^{\mathcal{L}}((\mathbf{id}_{\mathcal{E}}-C_{w^{\mathcal{E}}})^{-1}C_{w^{\mathcal{E}}}\mathrm{I})(z)\mathcal{X}^{\natural}(z)}{(z-\overline{\varsigma_n})(z-\zeta)}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\right),\quad \zeta\in\mathbb{C}\setminus\Sigma_{\mathcal{E}}.$$

Using the Cauchy-Schwarz inequality for integrals, one arrives at

$$|\mathcal{E}(\zeta)-\mathrm{I}|\leqslant\sum_{n=m+1}^{N}\left(\frac{|C_n^{\mathcal{K}}|}{2\pi}||\mathcal{X}^{\flat}(\cdot)||_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widehat{\mathcal{K}}_n)}\left|\left|\frac{\mathrm{I}}{(\cdot-\varsigma_n)(\cdot-\zeta)}\right|\right|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widehat{\mathcal{K}}_n)}+\frac{|C_n^{\mathcal{L}}|}{2\pi}||\mathcal{X}^{\natural}(\cdot)||_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widehat{\mathcal{L}}_n)}\right.$$
$$\times\left|\left|\frac{\mathrm{I}}{(\cdot-\overline{\varsigma_n})(\cdot-\zeta)}\right|\right|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widehat{\mathcal{L}}_n)}+\frac{|C_n^{\mathcal{K}}|}{2\pi}||(\mathbf{id}_{\mathcal{E}}-C_{w^{\mathcal{E}}})^{-1}||_{\mathcal{N}(\widehat{\mathcal{K}}_n)}||(C_{w^{\mathcal{E}}}\mathrm{I})(\cdot)||_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widehat{\mathcal{K}}_n)}$$
$$\times||\mathcal{X}^{\flat}(\cdot)||_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widehat{\mathcal{K}}_n)}\left|\left|\frac{\mathrm{I}}{(\cdot-\varsigma_n)(\cdot-\zeta)}\right|\right|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widehat{\mathcal{K}}_n)}+\frac{|C_n^{\mathcal{L}}|}{2\pi}||(\mathbf{id}_{\mathcal{E}}-C_{w^{\mathcal{E}}})^{-1}||_{\mathcal{N}(\widehat{\mathcal{L}}_n)}$$
$$\left.\times||(C_{w^{\mathcal{E}}}\mathrm{I})(\cdot)||_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widehat{\mathcal{L}}_n)}||\mathcal{X}^{\natural}(\cdot)||_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widehat{\mathcal{L}}_n)}\left|\left|\frac{\mathrm{I}}{(\cdot-\overline{\varsigma_n})(\cdot-\zeta)}\right|\right|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widehat{\mathcal{L}}_n)}\right),\quad\zeta\in\mathbb{C}\setminus\Sigma_{\mathcal{E}}.$$

One shows that, for $\zeta\in\mathbb{C}\setminus\Sigma_{\mathcal{E}}$,

$$\left|\left|\frac{\mathrm{I}}{(\cdot-\varsigma_n)(\cdot-\zeta)}\right|\right|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widehat{\mathcal{K}}_n)}\leqslant\sqrt{\frac{2}{\widehat{\varepsilon}_n^{\mathcal{K}}}}\left(\int_0^{2\pi}\frac{\mathrm{d}\omega}{|\zeta-\widehat{\varepsilon}_n^{\mathcal{K}}\mathrm{e}^{-\mathrm{i}\omega}|^2}\right)^{1/2}=:\sqrt{\frac{2}{\widehat{\varepsilon}_n^{\mathcal{K}}}}\,\mathcal{F}_{\widehat{\mathcal{K}}_n}(\zeta),$$
$$\left|\left|\frac{\mathrm{I}}{(\cdot-\overline{\varsigma_n})(\cdot-\zeta)}\right|\right|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widehat{\mathcal{L}}_n)}\leqslant\sqrt{\frac{2}{\widehat{\varepsilon}_n^{\mathcal{L}}}}\left(\int_0^{2\pi}\frac{\mathrm{d}\omega}{|\zeta-\widehat{\varepsilon}_n^{\mathcal{L}}\mathrm{e}^{\mathrm{i}\omega}|^2}\right)^{1/2}=:\sqrt{\frac{2}{\widehat{\varepsilon}_n^{\mathcal{L}}}}\,\mathcal{F}_{\widehat{\mathcal{L}}_n}(\zeta),$$

with $\mathcal{F}_{\widehat{\star}_n}(\zeta)=_{\zeta\to\infty}\mathcal{O}(|\zeta|^{-1})$ and $\mathcal{F}_{\widehat{\star}_n}(\zeta)=_{\zeta\to 0}\mathcal{O}(1)$, $\star\in\{\mathcal{K},\mathcal{L}\}$. Again, via the Cauchy-Schwarz inequality for integrals,

$$\begin{aligned}||(C_{w^{\mathcal{E}}}\mathrm{I})(\cdot)||_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widehat{\mathcal{K}}_n)}&\leqslant||(C_-(\mathrm{I}w_+^{\mathcal{E}}))(\cdot)||_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widehat{\mathcal{K}}_n)}\leqslant||C_-||_{\mathcal{N}(\widehat{\mathcal{K}}_n)}||w_+^{\mathcal{E}}(\cdot)||_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widehat{\mathcal{K}}_n)}\\&\leqslant||C_-||_{\mathcal{N}(\widehat{\mathcal{K}}_n)}\left|\left|\frac{C_n^{\mathcal{K}}}{(\cdot-\varsigma_n)}\mathcal{X}^{\flat}(\cdot)\right|\right|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widehat{\mathcal{K}}_n)}\\&\leqslant||C_-||_{\mathcal{N}(\widehat{\mathcal{K}}_n)}|C_n^{\mathcal{K}}|||\mathcal{X}^{\flat}(\cdot)||_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widehat{\mathcal{K}}_n)}\left|\left|\frac{\mathrm{I}}{(\cdot-\varsigma_n)}\right|\right|_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widehat{\mathcal{K}}_n)}\\&\leqslant 2\sqrt{\frac{\pi}{\widehat{\varepsilon}_n^{\mathcal{K}}}}\,|C_n^{\mathcal{K}}|||C_-||_{\mathcal{N}(\widehat{\mathcal{K}}_n)}||\mathcal{X}^{\flat}(\cdot)||_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widehat{\mathcal{K}}_n)},\end{aligned}$$

with an analogous estimate for $||(C_{w^{\mathcal{E}}}\mathrm{I})(\cdot)||_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widehat{\mathcal{L}}_n)}$:

$$||(C_{w^{\mathcal{E}}}\mathrm{I})(\cdot)||_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widehat{\mathcal{L}}_n)}\leqslant 2\sqrt{\frac{\pi}{\widehat{\varepsilon}_n^{\mathcal{L}}}}\,|C_n^{\mathcal{L}}|||C_-||_{\mathcal{N}(\widehat{\mathcal{L}}_n)}||\mathcal{X}^{\natural}(\cdot)||_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widehat{\mathcal{L}}_n)}.$$

Hence, for $\zeta\in\mathbb{C}\setminus\Sigma_{\mathcal{E}}$,

$$|\mathcal{E}(\zeta)-\mathrm{I}|\leqslant\sum_{n=m+1}^{N}\left(\frac{|C_n^{\mathcal{K}}|\mathcal{F}_{\widehat{\mathcal{K}}_n}(\zeta)}{\pi\sqrt{2\widehat{\varepsilon}_n^{\mathcal{K}}}}||\mathcal{X}^{\flat}(\cdot)||_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widehat{\mathcal{K}}_n)}+\frac{|C_n^{\mathcal{L}}|\mathcal{F}_{\widehat{\mathcal{L}}_n}(\zeta)}{\pi\sqrt{2\widehat{\varepsilon}_n^{\mathcal{L}}}}||\mathcal{X}^{\natural}(\cdot)||_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widehat{\mathcal{L}}_n)}\right.$$
$$+\frac{\sqrt{2}\,|C_n^{\mathcal{K}}|^2\mathcal{F}_{\widehat{\mathcal{K}}_n}(\zeta)}{\sqrt{\pi}\,\widehat{\varepsilon}_n^{\mathcal{K}}}||(\mathbf{id}_{\mathcal{E}}-C_{w^{\mathcal{E}}})^{-1}||_{\mathcal{N}(\widehat{\mathcal{K}}_n)}||C_-||_{\mathcal{N}(\widehat{\mathcal{K}}_n)}||\mathcal{X}^{\flat}(\cdot)||^2_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widehat{\mathcal{K}}_n)}$$
$$\left.+\frac{\sqrt{2}\,|C_n^{\mathcal{L}}|^2\mathcal{F}_{\widehat{\mathcal{L}}_n}(\zeta)}{\sqrt{\pi}\,\widehat{\varepsilon}_n^{\mathcal{L}}}||(\mathbf{id}_{\mathcal{E}}-C_{w^{\mathcal{E}}})^{-1}||_{\mathcal{N}(\widehat{\mathcal{L}}_n)}||C_-||_{\mathcal{N}(\widehat{\mathcal{L}}_n)}||\mathcal{X}^{\natural}(\cdot)||^2_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widehat{\mathcal{L}}_n)}\right).$$

It is shown, *a posteriori*, in Section 4 that the RHP for $\widehat{\chi}(\zeta)$ formulated in the Lemma is asymptotically solvable, whence $||\mathcal{X}^{\flat}(\cdot)||^2_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widehat{\mathcal{K}}_n)}\leqslant\text{const.}=\underline{c}$ and $||\mathcal{X}^{\natural}(\cdot)||^2_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widehat{\mathcal{L}}_n)}\leqslant\text{const.}=\underline{c}$. Furthermore [38], $||(\mathbf{id}_{\mathcal{E}}-C_{w^{\mathcal{E}}})^{-1}||_{\mathcal{N}(\widehat{\star}_n)}\leqslant\text{const.}\,||(\mathbf{id}_{\mathcal{E}}-C_{w^{\mathcal{E}}})^{-1}||_{\mathcal{N}(\Sigma_{\mathcal{E}})}\leqslant\underline{c}$ (see above), $\star\in\{\mathcal{K},\mathcal{L}\}$. Recalling

the expressions for $C_n^{\mathcal{K}}$ and $C_n^{\mathcal{L}}$ given in Lemma 3.3, that as $t\to+\infty$ and $x\to-\infty$ such that $z_o:=x/t<-2$ and $(x,t)\in\daleth_m$, $(g_n)^{-1}=\mathcal{O}(\exp(-4t\sin(\phi_n)|\cos(\phi_n)-\cos(\phi_m)|))$, and the definition $||\mathcal{E}(\cdot)-\mathrm{I}||_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\mathbb{C}\setminus\Sigma_{\mathcal{E}})}:=\max_{i,j\in\{1,2\}}\sup_{\zeta\in\mathbb{C}\setminus\Sigma_{\mathcal{E}}}|(\mathcal{E}(\zeta)-\mathrm{I})_{ij}|$, assembling the above, one arrives at

$$||\mathcal{E}(\cdot)-\mathrm{I}||_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\mathbb{C}\setminus\Sigma_{\mathcal{E}})}\leqslant\mathcal{O}\left(\mathcal{F}_{\mathcal{E}}(\zeta)\exp\left(-4t\min_{\substack{m\in\{1,2,\dots,N\}\\ n\in\{m+1,m+2,\dots,N\}}}\{\sin(\phi_n)|\cos(\phi_n)-\cos(\phi_m)|\}\right)\right),$$

where $\mathcal{F}_{\mathcal{E}}(\zeta)=_{\zeta\to\infty}\mathcal{O}(|\zeta|^{-1})$ and $\mathcal{F}_{\mathcal{E}}(\zeta)=_{\zeta\to 0}\mathcal{O}(1)$; hence, the asymptotic estimate for $\widehat{m}^{\sharp}(\zeta)$ stated in the Lemma. Finally, from the asymptotics for $\mathcal{E}(\zeta)-\mathrm{I}$ derived above, the ordered factorisation for $\widehat{m}^{\sharp}(\zeta)$ given in Proposition 3.1, and Eqs. (61) and (62), the large-ζ asymptotics lead one to Eqs. (63) and (64). □

4 Asymptotic Solution of the Model RHP

In this section, the model (normalised at ∞) RHP for $\widehat{\chi}(\zeta)$ formulated in Lemma 3.5 is solved asymptotically as $t\to+\infty$ and $x\to-\infty$ such that $z_o:=x/t<-2$ and $(x,t)\in\daleth_m$, $m\in\{1,2,\dots,N\}$, and the corresponding (asymptotic) results for $u(x,t)$, the solution of the Cauchy problem for the D_fNLSE, and $\int_{\pm\infty}^{x}(|u(x',t)|^2-1)\,\mathrm{d}x'$ stated in Theorem 2.2.1 (for the upper sign) are derived.

Lemma 4.1. *The solution of the* RHP *for* $\widehat{\chi}(\zeta)\colon\mathbb{C}\setminus(\widehat{\sigma}_d\cup\sigma_c)\to\mathrm{M}_2(\mathbb{C})$ *formulated in Lemma* 3.5 *is given by the following ordered factorisation,*

$$\widehat{\chi}(\zeta)=\left(\mathrm{I}+\zeta^{-1}\widehat{\Delta}_o\right)\widehat{\mathcal{P}}(\zeta)\widehat{m}_d(\zeta)\chi^c(\zeta),\quad \zeta\in\mathbb{C}\setminus(\widehat{\sigma}_d\cup\sigma_c),$$

where $\widehat{m}_d(\zeta)=\sigma_1\overline{\widehat{m}_d(\overline{\zeta})}\,\sigma_1$ $(\in\mathrm{SL}(2,\mathbb{C}))$ *has the representation*

$$\widehat{m}_d(\zeta)=\mathrm{I}+\sum_{n=1}^{m}\left(\frac{\mathrm{Res}(\widehat{\chi}(\zeta);\varsigma_n)}{(\zeta-\varsigma_n)}+\frac{\sigma_1\overline{\mathrm{Res}(\widehat{\chi}(\zeta);\varsigma_n)}\,\sigma_1}{(\zeta-\overline{\varsigma_n})}\right),$$

$\widehat{\mathcal{P}}(\zeta)=\sigma_1\overline{\widehat{\mathcal{P}}(\overline{\zeta})}\,\sigma_1$ *is chosen (see Lemma* 4.3 *below) so that* $\widehat{\Delta}_o$ *is idempotent,* $\mathrm{I}+\zeta^{-1}\widehat{\Delta}_o$ *is holomorphic in a punctured neighbourhood of the origin, with* $\widehat{\Delta}_o=\sigma_1\overline{\widehat{\Delta}_o}\,\sigma_1$ $(\in\mathrm{GL}(2,\mathbb{C}))$ *and* $\det(\mathrm{I}+\zeta^{-1}\widehat{\Delta}_o)|_{\zeta=\pm 1}=0$, *and determined by the relation*

$$\widehat{\Delta}_o=\widehat{\mathcal{P}}(0)\widehat{m}_d(0)\chi^c(0)(\delta(0))^{\sigma_3}\left(\prod_{k=m+1}^{N}(d_k^+(0))^{\sigma_3}\right)\sigma_2,$$

and satisfying $\mathrm{tr}(\widehat{\Delta}_o)=0$, $\det(\widehat{\Delta}_o)=-1$, *and* $\widehat{\Delta}_o\widehat{\Delta}_o=\mathrm{I}$, *and* $\chi^c(\zeta)\colon\mathbb{C}\setminus\sigma_c\to\mathrm{SL}(2,\mathbb{C})$ *solves the following* RHP: (1) $\chi^c(\zeta)$ *is piecewise (sectionally) holomorphic* $\forall\,\zeta\in\mathbb{C}\setminus\sigma_c$; (2) $\chi^c_{\pm}(\zeta):=\lim_{\substack{\zeta'\to\zeta\\ \pm\mathrm{Im}(\zeta')>0}}\chi^c(\zeta')$ *satisfy, for* $\zeta\in\mathbb{R}$, *the jump condition*

$$\chi^c_+(\zeta)=\chi^c_-(\zeta)\mathrm{e}^{-\mathrm{i}k(\zeta)(x+2\lambda(\zeta)t)\mathrm{ad}(\sigma_3)}\begin{pmatrix}(1-r(\zeta)\overline{r(\overline{\zeta})})\delta_-(\zeta)/\delta_+(\zeta) & -\frac{\overline{r(\overline{\zeta})}}{(\delta_-(\zeta)\delta_+(\zeta))^{-1}}\prod_{k=m+1}^{N}(d_k^+(\zeta))^2\\ \frac{r(\zeta)}{\delta_-(\zeta)\delta_+(\zeta)}\prod_{k=m+1}^{N}(d_k^+(\zeta))^{-2} & \delta_+(\zeta)/\delta_-(\zeta)\end{pmatrix};$$

(3) $\chi^c(\zeta)\underset{\substack{\zeta\to\infty\\ \zeta\in\mathbb{C}\setminus\sigma_c}}{=}\mathrm{I}+\mathcal{O}(\zeta^{-1})$; *and* (4) $\chi^c(\zeta)=\sigma_1\overline{\chi^c(\overline{\zeta})}\,\sigma_1$.

Proof. One verifies that, modulo the explicit determination of $\widehat{\Delta}_o$, $\widehat{\mathcal{P}}(\zeta)$, $\widehat{m}_d(\zeta)$, and $\chi^c(\zeta)$, the ordered factorisation for $\widehat{\chi}(\zeta)$ stated in the Lemma, with the conditions on $\widehat{\Delta}_o$, $\widehat{\mathcal{P}}(\zeta)$, $\widehat{m}_d(\zeta)$, and $\chi^c(\zeta)$ stated therein, solves the RHP for $\widehat{\chi}(\zeta)$ stated in Lemma 3.5. □

The determination of the asymptotics for the solution of the RHP for $\chi^c(\zeta)\colon\mathbb{C}\setminus\sigma_c\to\mathrm{SL}(2,\mathbb{C})$ stated in Lemma 4.1 was the (principal) subject of study in [38], and is given by the following

Lemma 4.2. *Let* ε *be an arbitrarily fixed, sufficiently small positive real number, and, for* $z\in\{\lambda_1,\lambda_2\}$, *with* λ_1 *and* λ_2 *given in Theorem* 2.2.1, *Eq.* (10), *set* $\mathbb{U}(z;\varepsilon):=\{\zeta;\,|\zeta-z|<\varepsilon\}$. *Then, as* $t\to+\infty$ *and*

$x \to -\infty$ such that $z_o := x/t < -2$, for $\zeta \in \mathbb{C} \setminus \cup_{z\in\{\lambda_1,\lambda_2\}} \mathbb{U}(z;\varepsilon)$, $\chi^c(\zeta)$ has the following asymptotics:

$$\chi^c_{11}(\zeta) = 1 + \mathcal{O}\left(\left(\frac{c^{\mathcal{S}}(\lambda_1)\underline{c}(\lambda_2,\lambda_3,\overline{\lambda_3})}{\sqrt{\lambda_2(z_o^2+32)}\,(\zeta-\lambda_1)} + \frac{c^{\mathcal{S}}(\lambda_2)\underline{c}(\lambda_1,\lambda_3,\overline{\lambda_3})}{\sqrt{\lambda_1(z_o^2+32)}\,(\zeta-\lambda_2)}\right)\frac{\ln t}{(\lambda_1-\lambda_2)t}\right),$$

$$\begin{aligned}\chi^c_{12}(\zeta) = \mathrm{e}^{\frac{\mathrm{i}\Xi^+(0)}{2}}\Bigg(&\frac{\sqrt{\nu(\lambda_1)}\,\lambda_1^{2\mathrm{i}\nu(\lambda_1)}}{\sqrt{t(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\left(\frac{\lambda_1\mathrm{e}^{-\mathrm{i}(\Theta^+(z_o,t)+\frac{\pi}{4})}}{(\zeta-\lambda_1)} + \frac{\lambda_2\mathrm{e}^{\mathrm{i}(\Theta^+(z_o,t)+\frac{\pi}{4})}}{(\zeta-\lambda_2)}\right)\\ &+ \mathcal{O}\left(\left(\frac{c^{\mathcal{S}}(\lambda_1)\underline{c}(\lambda_2,\lambda_3,\overline{\lambda_3})}{\sqrt{\lambda_2(z_o^2+32)}\,(\zeta-\lambda_1)} + \frac{c^{\mathcal{S}}(\lambda_2)\underline{c}(\lambda_1,\lambda_3,\overline{\lambda_3})}{\sqrt{\lambda_1(z_o^2+32)}\,(\zeta-\lambda_2)}\right)\frac{\ln t}{(\lambda_1-\lambda_2)t}\right)\Bigg),\end{aligned}$$

$$\begin{aligned}\chi^c_{21}(\zeta) = \mathrm{e}^{-\frac{\mathrm{i}\Xi^+(0)}{2}}\Bigg(&\frac{\sqrt{\nu(\lambda_1)}\,\lambda_1^{-2\mathrm{i}\nu(\lambda_1)}}{\sqrt{t(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\left(\frac{\lambda_1\mathrm{e}^{\mathrm{i}(\Theta^+(z_o,t)+\frac{\pi}{4})}}{(\zeta-\lambda_1)} + \frac{\lambda_2\mathrm{e}^{-\mathrm{i}(\Theta^+(z_o,t)+\frac{\pi}{4})}}{(\zeta-\lambda_2)}\right)\\ &+ \mathcal{O}\left(\left(\frac{c^{\mathcal{S}}(\lambda_1)\underline{c}(\lambda_2,\lambda_3,\overline{\lambda_3})}{\sqrt{\lambda_2(z_o^2+32)}\,(\zeta-\lambda_1)} + \frac{c^{\mathcal{S}}(\lambda_2)\underline{c}(\lambda_1,\lambda_3,\overline{\lambda_3})}{\sqrt{\lambda_1(z_o^2+32)}\,(\zeta-\lambda_2)}\right)\frac{\ln t}{(\lambda_1-\lambda_2)t}\right)\Bigg),\end{aligned}$$

$$\chi^c_{22}(\zeta) = 1 + \mathcal{O}\left(\left(\frac{c^{\mathcal{S}}(\lambda_1)\underline{c}(\lambda_2,\lambda_3,\overline{\lambda_3})}{\sqrt{\lambda_2(z_o^2+32)}\,(\zeta-\lambda_1)} + \frac{c^{\mathcal{S}}(\lambda_2)\underline{c}(\lambda_1,\lambda_3,\overline{\lambda_3})}{\sqrt{\lambda_1(z_o^2+32)}\,(\zeta-\lambda_2)}\right)\frac{\ln t}{(\lambda_1-\lambda_2)t}\right),$$

where λ_3, $\nu(\cdot)$, $\Theta^+(z_o,t)$, and $\Xi^+(\cdot)$, respectively, are given in Theorem 2.2.1, Eqs. (10), (11), (17), *and* (18), $||(\cdot-\lambda_k)^{-1}||_{\mathcal{L}^\infty(\mathbb{C}\setminus\cup_{z\in\{\lambda_1,\lambda_2\}}\mathbb{U}(z;\varepsilon))} < \infty$, $k \in \{1,2\}$, $\chi^c(\zeta) = \sigma_1\overline{\chi^c(\overline{\zeta})}\,\sigma_1$, *and* $(\chi^c(0)\sigma_2)^2 = \mathrm{I}$ $(+\,\mathcal{O}(t^{-1}\ln t))$.

Sketch of Proof. Proceeding as in the proof of Lemma 6.1 in [38] and particularising it to the case of the RHP for $\chi^c(\zeta)$ stated in Lemma 4.1, one arrives at

$$\begin{aligned}\chi^c_{11}(\zeta) = 1 &- \frac{\widehat{r}(\lambda_1)(\delta_B^0)^{-2}\mathrm{e}^{\frac{\pi\nu}{2}}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}}{2\pi\mathrm{i}(\zeta-\lambda_1)\beta_{21}^{\Sigma_{B^0}}\mathcal{X}_B\sqrt{t}}\int_0^{+\infty}(\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}\partial_z\mathbf{D}_{-\mathrm{i}\nu}(z) - \tfrac{\mathrm{i}}{2}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}z\mathbf{D}_{-\mathrm{i}\nu}(z))z^{-\mathrm{i}\nu}\mathrm{e}^{-\frac{z^2}{4}}\,\mathrm{d}z\\ &+ \frac{\widehat{r}(\lambda_1)(1-|\widehat{r}(\lambda_1)|^2)^{-1}(\delta_B^0)^{-2}\mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}}{2\pi\mathrm{i}(\zeta-\lambda_1)\beta_{21}^{\Sigma_{B^0}}\mathrm{e}^{\frac{3\pi\nu}{2}}\mathcal{X}_B\sqrt{t}}\int_0^{+\infty}(\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}\partial_z\mathbf{D}_{-\mathrm{i}\nu}(z) - \tfrac{\mathrm{i}}{2}\mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}z\mathbf{D}_{-\mathrm{i}\nu}(z))z^{-\mathrm{i}\nu}\mathrm{e}^{-\frac{z^2}{4}}\,\mathrm{d}z\\ &- \frac{\overline{\widehat{r}(\lambda_1)}(\delta_A^0)^{-2}\mathrm{e}^{-\frac{\pi\nu}{2}}(-1)^{\mathrm{i}\nu}\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}}{2\pi\mathrm{i}(\zeta-\lambda_2)\beta_{21}^{\Sigma_{A^0}}\mathcal{X}_A\sqrt{t}}\int_0^{+\infty}(\mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}\partial_z\mathbf{D}_{\mathrm{i}\nu}(z) + \tfrac{\mathrm{i}}{2}\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}z\mathbf{D}_{\mathrm{i}\nu}(z))z^{\mathrm{i}\nu}\mathrm{e}^{-\frac{z^2}{4}}\,\mathrm{d}z\\ &+ \frac{\overline{\widehat{r}(\lambda_1)}(1-|\widehat{r}(\lambda_1)|^2)^{-1}(\delta_A^0)^{-2}(-1)^{\mathrm{i}\nu}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}}{2\pi\mathrm{i}(\zeta-\lambda_2)\beta_{21}^{\Sigma_{A^0}}\mathrm{e}^{\frac{\pi\nu}{2}}\mathcal{X}_A\sqrt{t}}\int_0^{+\infty}(\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\partial_z\mathbf{D}_{\mathrm{i}\nu}(z) + \tfrac{\mathrm{i}}{2}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}z\mathbf{D}_{\mathrm{i}\nu}(z))z^{\mathrm{i}\nu}\mathrm{e}^{-\frac{z^2}{4}}\,\mathrm{d}z\\ &+ \mathcal{O}\left(\left(\frac{c^{\mathcal{S}}(\lambda_1)\underline{c}(\lambda_2,\lambda_3,\overline{\lambda_3})(\delta_B^0)^{-2}}{(\zeta-\lambda_1)|\lambda_1-\lambda_3|\sqrt{(\lambda_1-\lambda_2)}\;\mathcal{X}_B} + \frac{c^{\mathcal{S}}(\lambda_2)\underline{c}(\lambda_1,\lambda_3,\overline{\lambda_3})(\delta_A^0)^{-2}}{(\zeta-\lambda_2)|\lambda_2-\lambda_3|\sqrt{(\lambda_1-\lambda_2)}\;\mathcal{X}_A}\right)\frac{\ln t}{t}\right),\end{aligned}$$

$$\begin{aligned}\chi^c_{12}(\zeta) = &\left(\frac{\overline{\widehat{r}(\lambda_1)}(\delta_B^0)^{2}\mathrm{e}^{\frac{\pi\nu}{2}}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}}{2\pi\mathrm{i}(\zeta-\lambda_1)\mathcal{X}_B\sqrt{t}} - \frac{\overline{\widehat{r}(\lambda_1)}(1-|\widehat{r}(\lambda_1)|^2)^{-1}(\delta_B^0)^{2}\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}}{2\pi\mathrm{i}(\zeta-\lambda_1)\mathrm{e}^{\frac{3\pi\nu}{2}}\mathcal{X}_B\sqrt{t}}\right)\int_0^{+\infty}\mathbf{D}_{\mathrm{i}\nu}(z)z^{\mathrm{i}\nu}\mathrm{e}^{-\frac{z^2}{4}}\,\mathrm{d}z\\ &+ \left(\frac{\widehat{r}(\lambda_1)(\delta_A^0)^{2}\mathrm{e}^{-\frac{\pi\nu}{2}}\mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}}{2\pi\mathrm{i}(\zeta-\lambda_2)(-1)^{\mathrm{i}\nu}\mathcal{X}_A\sqrt{t}} - \frac{\widehat{r}(\lambda_1)(1-|\widehat{r}(\lambda_1)|^2)^{-1}(\delta_A^0)^{2}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}}{2\pi\mathrm{i}(\zeta-\lambda_2)\mathrm{e}^{\frac{\pi\nu}{2}}(-1)^{\mathrm{i}\nu}\mathcal{X}_A\sqrt{t}}\right)\int_0^{+\infty}\mathbf{D}_{-\mathrm{i}\nu}(z)z^{-\mathrm{i}\nu}\mathrm{e}^{-\frac{z^2}{4}}\,\mathrm{d}z\\ &+ \mathcal{O}\left(\left(\frac{c^{\mathcal{S}}(\lambda_1)\underline{c}(\lambda_2,\lambda_3,\overline{\lambda_3})(\delta_B^0)^{2}}{(\zeta-\lambda_1)|\lambda_1-\lambda_3|\sqrt{(\lambda_1-\lambda_2)}\;\mathcal{X}_B} + \frac{c^{\mathcal{S}}(\lambda_2)\underline{c}(\lambda_1,\lambda_3,\overline{\lambda_3})(\delta_A^0)^{2}}{(\zeta-\lambda_2)|\lambda_2-\lambda_3|\sqrt{(\lambda_1-\lambda_2)}\;\mathcal{X}_A}\right)\frac{\ln t}{t}\right),\end{aligned}$$

$$\begin{aligned}\chi^c_{21}(\zeta) = -&\left(\frac{\widehat{r}(\lambda_1)(\delta_B^0)^{-2}\mathrm{e}^{\frac{\pi\nu}{2}}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}}{2\pi\mathrm{i}(\zeta-\lambda_1)\mathcal{X}_B\sqrt{t}} - \frac{\widehat{r}(\lambda_1)(1-|\widehat{r}(\lambda_1)|^2)^{-1}(\delta_B^0)^{-2}\mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}}{2\pi\mathrm{i}(\zeta-\lambda_1)\mathrm{e}^{\frac{3\pi\nu}{2}}\mathcal{X}_B\sqrt{t}}\right)\int_0^{+\infty}\mathbf{D}_{-\mathrm{i}\nu}(z)z^{-\mathrm{i}\nu}\mathrm{e}^{-\frac{z^2}{4}}\,\mathrm{d}z\\ &- \left(\frac{\overline{\widehat{r}(\lambda_1)}(\delta_A^0)^{-2}\mathrm{e}^{-\frac{\pi\nu}{2}}\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}}{2\pi\mathrm{i}(\zeta-\lambda_2)(-1)^{-\mathrm{i}\nu}\mathcal{X}_A\sqrt{t}} - \frac{\overline{\widehat{r}(\lambda_1)}(1-|\widehat{r}(\lambda_1)|^2)^{-1}(\delta_A^0)^{-2}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}}{2\pi\mathrm{i}(\zeta-\lambda_2)\mathrm{e}^{\frac{\pi\nu}{2}}(-1)^{-\mathrm{i}\nu}\mathcal{X}_A\sqrt{t}}\right)\int_0^{+\infty}\mathbf{D}_{\mathrm{i}\nu}(z)z^{\mathrm{i}\nu}\mathrm{e}^{-\frac{z^2}{4}}\,\mathrm{d}z\\ &+ \mathcal{O}\left(\left(\frac{c^{\mathcal{S}}(\lambda_1)\underline{c}(\lambda_2,\lambda_3,\overline{\lambda_3})(\delta_B^0)^{-2}}{(\zeta-\lambda_1)|\lambda_1-\lambda_3|\sqrt{(\lambda_1-\lambda_2)}\;\mathcal{X}_B} + \frac{c^{\mathcal{S}}(\lambda_2)\underline{c}(\lambda_1,\lambda_3,\overline{\lambda_3})(\delta_A^0)^{-2}}{(\zeta-\lambda_2)|\lambda_2-\lambda_3|\sqrt{(\lambda_1-\lambda_2)}\;\mathcal{X}_A}\right)\frac{\ln t}{t}\right),\end{aligned}$$

$$\begin{aligned}\chi^c_{22}(\zeta) = 1 &+ \frac{\overline{\widehat{r}(\lambda_1)}(\delta_B^0)^{2}\mathrm{e}^{\frac{\pi\nu}{2}}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}}{2\pi\mathrm{i}(\zeta-\lambda_1)\beta_{12}^{\Sigma_{B^0}}\mathcal{X}_B\sqrt{t}}\int_0^{+\infty}(\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\partial_z\mathbf{D}_{\mathrm{i}\nu}(z) + \tfrac{\mathrm{i}}{2}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}z\mathbf{D}_{\mathrm{i}\nu}(z))z^{\mathrm{i}\nu}\mathrm{e}^{-\frac{z^2}{4}}\,\mathrm{d}z\\ &- \frac{\overline{\widehat{r}(\lambda_1)}(1-|\widehat{r}(\lambda_1)|^2)^{-1}(\delta_B^0)^{2}\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}}{2\pi\mathrm{i}(\zeta-\lambda_1)\beta_{12}^{\Sigma_{B^0}}\mathrm{e}^{\frac{3\pi\nu}{2}}\mathcal{X}_B\sqrt{t}}\int_0^{+\infty}(\mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}\partial_z\mathbf{D}_{\mathrm{i}\nu}(z) + \tfrac{\mathrm{i}}{2}\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}z\mathbf{D}_{\mathrm{i}\nu}(z))z^{\mathrm{i}\nu}\mathrm{e}^{-\frac{z^2}{4}}\,\mathrm{d}z\end{aligned}$$

$$
\begin{aligned}
&+\frac{\widehat{r}(\lambda_1)(\delta_A^0)^2\mathrm{e}^{-\frac{\pi\nu}{2}}\mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}}{2\pi\mathrm{i}(\zeta-\lambda_2)\beta_{12}^{\Sigma_{A^0}}(-1)^{\mathrm{i}\nu}\mathcal{X}_A\sqrt{t}}\int_0^{+\infty}(\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}\partial_z\mathbf{D}_{-\mathrm{i}\nu}(z)-\tfrac{\mathrm{i}}{2}\mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}z\mathbf{D}_{-\mathrm{i}\nu}(z))z^{-\mathrm{i}\nu}\mathrm{e}^{-\frac{z^2}{4}}\,\mathrm{d}z\\
&-\frac{\widehat{r}(\lambda_1)(1-|\widehat{r}(\lambda_1)|^2)^{-1}(\delta_A^0)^2\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}}{2\pi\mathrm{i}(\zeta-\lambda_2)\beta_{12}^{\Sigma_{A^0}}\mathrm{e}^{\frac{\pi\nu}{2}}(-1)^{\mathrm{i}\nu}\mathcal{X}_A\sqrt{t}}\int_0^{+\infty}(\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}\partial_z\mathbf{D}_{-\mathrm{i}\nu}(z)-\tfrac{\mathrm{i}}{2}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}z\mathbf{D}_{-\mathrm{i}\nu}(z))z^{-\mathrm{i}\nu}\mathrm{e}^{-\frac{z^2}{4}}\,\mathrm{d}z\\
&+\mathcal{O}\left(\left(\frac{c^{\mathcal{S}}(\lambda_1)\underline{c}(\lambda_2,\lambda_3,\overline{\lambda_3})(\delta_B^0)^2}{(\zeta-\lambda_1)|\lambda_1-\lambda_3|\sqrt{(\lambda_1-\lambda_2)}\,\mathcal{X}_B}+\frac{c^{\mathcal{S}}(\lambda_2)\underline{c}(\lambda_1,\lambda_3,\overline{\lambda_3})(\delta_A^0)^2}{(\zeta-\lambda_2)|\lambda_2-\lambda_3|\sqrt{(\lambda_1-\lambda_2)}\,\mathcal{X}_A}\right)\frac{\ln t}{t}\right),
\end{aligned}
$$

where $\widehat{r}(\zeta)=r(\zeta)\prod_{k=m+1}^{N}(d_k^+(\zeta))^{-2}$ $(|\widehat{r}(\lambda_1)|=|r(\lambda_1)|)$, $\nu=\nu(\lambda_1)$,

$$
\begin{gathered}
\delta_B^0=|\lambda_1-\lambda_3|^{-\mathrm{i}\nu}\big(2t(\lambda_1-\lambda_2)^3\lambda_1^{-3}\big)^{-\frac{\mathrm{i}\nu}{2}}\mathrm{e}^{\mathcal{Z}(\lambda_1)}\exp\big(-\tfrac{\mathrm{i}t}{2}(\lambda_1-\lambda_2)(z_o+\lambda_1+\lambda_2)\big),\\
\delta_A^0=|\lambda_2-\lambda_3|^{\mathrm{i}\nu}\big(2t(\lambda_1-\lambda_2)^3\lambda_2^{-3}\big)^{\frac{\mathrm{i}\nu}{2}}\mathrm{e}^{\mathcal{Z}(\lambda_2)}\exp\big(\tfrac{\mathrm{i}t}{2}(\lambda_1-\lambda_2)(z_o+\lambda_1+\lambda_2)\big),\\
\mathcal{Z}(\lambda_1)=\frac{\mathrm{i}}{2\pi}\int_{-\infty}^{0}\ln|\mu-\lambda_1|\,\mathrm{d}\ln(1-|r(\mu)|^2)+\frac{\mathrm{i}}{2\pi}\int_{\lambda_2}^{\lambda_1}\ln|\mu-\lambda_1|\,\mathrm{d}\ln(1-|r(\mu)|^2),\\
\mathcal{Z}(\lambda_2)=-\mathcal{Z}(\lambda_1)+\frac{\mathrm{i}}{2\pi}\int_{-\infty}^{0}\ln|\mu|\,\mathrm{d}\ln(1-|r(\mu)|^2)+\frac{\mathrm{i}}{2\pi}\int_{\lambda_2}^{\lambda_1}\ln|\mu|\,\mathrm{d}\ln(1-|r(\mu)|^2),\\
\mathcal{X}_B=\frac{|\lambda_1-\lambda_3|}{\lambda_1}\sqrt{\frac{2(\lambda_1-\lambda_2)}{\lambda_1}},\qquad \mathcal{X}_A=\frac{|\lambda_2-\lambda_3|}{\lambda_2}\sqrt{\frac{2(\lambda_1-\lambda_2)}{\lambda_2}},\\
\beta_{12}^{\Sigma_{B^0}}=\overline{\beta_{21}^{\Sigma_{B^0}}}=\frac{\sqrt{2\pi}\,\mathrm{e}^{-\frac{\pi\nu}{2}}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}}{\widehat{r}(\lambda_1)\,\Gamma(\mathrm{i}\nu)},\qquad \beta_{12}^{\Sigma_{A^0}}=\overline{\beta_{21}^{\Sigma_{A^0}}}=\frac{\sqrt{2\pi}\,\mathrm{e}^{-\frac{\pi\nu}{2}}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}}{\overline{\widehat{r}(\lambda_1)}\,\overline{\Gamma(\mathrm{i}\nu)}},
\end{gathered}
$$

$\Gamma(\cdot)$ is the gamma function [51], and $\mathbf{D}_*(\cdot)$ is the parabolic cylinder function [51]. Using Eq. (10), one shows that $|\lambda_k-\lambda_3|\lambda_k^{-1}=(2\lambda_k)^{-1/2}(z_o^2+32)^{1/4}$, $k\in\{1,2\}$. Using the identities [51] $\partial_z\mathbf{D}_{z_1}(z)=\frac{1}{2}(z_1\mathbf{D}_{z_1-1}(z)-\mathbf{D}_{z_1+1}(z))$, $z\mathbf{D}_{z_1}(z)=\mathbf{D}_{z_1+1}(z)+z_1\mathbf{D}_{z_1-1}(z)$, and $|\Gamma(\mathrm{i}\nu)|^2=\frac{\pi}{\nu\sinh(\pi\nu)}$, and the integral [51] $\int_0^{+\infty}\mathbf{D}_{-z_1}(z)z^{z_2-1}\mathrm{e}^{-z^2/4}\,\mathrm{d}z=\frac{\sqrt{\pi}\,\exp\left(-\frac{1}{2}(z_1+z_2)\ln 2\right)\Gamma(z_2)}{\Gamma(\frac{1}{2}(z_1+z_2)+\frac{1}{2})}$, $\mathrm{Re}(z_2)>0$, from the above expressions for $\chi_{ij}^c(\zeta)$, $i,j\in\{1,2\}$, and repeated application of the relation $|r(\lambda_1)||\Gamma(\mathrm{i}\nu)|\nu\mathrm{e}^{\frac{\pi\nu}{2}}=(2\pi\nu)^{1/2}$, one obtains the result stated in the Lemma. Furthermore, one shows that the symmetry reduction $\chi^c(\zeta)=\sigma_1\overline{\chi^c(\overline{\zeta})}\,\sigma_1$ is satisfied, and verifies that $(\chi^c(0)\sigma_2)^2=\mathrm{I}+\mathcal{O}(t^{-1}\ln t)$. □

Proposition 4.1. *For $m\in\{1,2,\ldots,N\}$, set $\mathrm{Res}(\widehat{\chi}(\zeta);\varsigma_n):=\left(\begin{smallmatrix}a_n & b_n\\ c_n & d_n\end{smallmatrix}\right)$, $n\in\{1,2,\ldots,m\}$. Then $b_n=-a_n\chi_{12}^c(\varsigma_n)/\chi_{22}^c(\varsigma_n)$, $d_n=-c_n\chi_{12}^c(\varsigma_n)/\chi_{22}^c(\varsigma_n)$, and $\{a_n,\overline{c_n}\}_{n=1}^m$ satisfy the following (non-singular) system of $2m$ linear inhomogeneous algebraic equations,*

$$
\begin{bmatrix}
\boxed{\widehat{\mathcal{A}}} & \boxed{\widehat{\mathcal{B}}}\\
\boxed{\overline{\widehat{\mathcal{B}}}} & \boxed{\overline{\widehat{\mathcal{A}}}}
\end{bmatrix}
\begin{bmatrix}
a_1\\ a_2\\ \vdots\\ a_m\\ \overline{c_1}\\ \overline{c_2}\\ \vdots\\ \overline{c_m}
\end{bmatrix}
=
\begin{bmatrix}
g_1^*\chi_{12}^c(\varsigma_1)\\ g_2^*\chi_{12}^c(\varsigma_2)\\ \vdots\\ g_m^*\chi_{12}^c(\varsigma_m)\\ \overline{g_1^*\chi_{22}^c(\varsigma_1)}\\ \overline{g_2^*\chi_{22}^c(\varsigma_2)}\\ \vdots\\ \overline{g_m^*\chi_{22}^c(\varsigma_m)}
\end{bmatrix},
$$

where

$$
\begin{gathered}
\widehat{\mathcal{A}}_{ij}:=\begin{cases}
\dfrac{\det(\chi^c(\varsigma_i))+g_i^*\mathrm{W}(\chi_{12}^c(\varsigma_i),\chi_{22}^c(\varsigma_i))}{\chi_{22}^c(\varsigma_i)}, & i=j\in\{1,2,\ldots,m\},\\[2mm]
-\dfrac{g_i^*(\chi_{12}^c(\varsigma_i)\chi_{22}^c(\varsigma_j)-\chi_{22}^c(\varsigma_i)\chi_{12}^c(\varsigma_j))}{(\varsigma_i-\varsigma_j)\chi_{22}^c(\varsigma_j)}, & i\neq j\in\{1,2,\ldots,m\},
\end{cases}\\
\widehat{\mathcal{B}}_{ij}:=-\frac{g_i^*(\chi_{22}^c(\varsigma_i)\overline{\chi_{22}^c(\varsigma_j)}-\chi_{12}^c(\varsigma_i)\overline{\chi_{12}^c(\varsigma_j)})}{(\varsigma_i-\overline{\varsigma_j})\overline{\chi_{22}^c(\varsigma_j)}},\qquad i,j\in\{1,2,\ldots,m\},
\end{gathered}
$$

$$
g_j^*=|g_j|\mathrm{e}^{\mathrm{i}\theta_{g_j}}\exp(2\mathrm{i}k(\varsigma_j)(x+2\lambda(\varsigma_j)t))(\delta(\varsigma_j))^{-2}\prod_{k=m+1}^{N}(d_k^+(\varsigma_j))^{-2},\quad j\in\{1,2,\ldots,m\},
$$

with $|g_j|$ and θ_{g_j} given in Lemma 3.1, (iii), and $\mathrm{W}(\chi_{12}^c(z),\chi_{22}^c(z))=\left|\begin{smallmatrix}\chi_{12}^c(z) & \chi_{22}^c(z)\\ \partial_z\chi_{12}^c(z) & \partial_z\chi_{22}^c(z)\end{smallmatrix}\right|$.

Proof. Recall from Lemma 4.1 that $\widehat{\chi}(\zeta)\colon \mathbb{C}\setminus(\widehat{\sigma}_d\cup\sigma_c)\to \mathrm{M}_2(\mathbb{C})$ has the factorised representation $\widehat{\chi}(\zeta)=\left(\mathrm{I}+\zeta^{-1}\widehat{\Delta}_o\right)\widehat{\mathcal{P}}(\zeta)\left(\mathrm{I}+\sum_{n=1}^{m}\left(\frac{\mathrm{Res}(\widehat{\chi}(\zeta);\varsigma_n)}{(\zeta-\varsigma_n)}+\frac{\sigma_1\overline{\mathrm{Res}(\widehat{\chi}(\zeta);\varsigma_n)}\,\sigma_1}{(\zeta-\overline{\varsigma_n})}\right)\right)\chi^c(\zeta)$, where $\chi^c(\zeta)$ is given in Lemma 4.2. For $m\in\{1,2,\dots,N\}$, set $\mathrm{Res}(\widehat{\chi}(\zeta);\varsigma_n):=\left(\begin{smallmatrix} a_n & b_n\\ c_n & d_n\end{smallmatrix}\right)$, whence $\sigma_1\overline{\mathrm{Res}(\widehat{\chi}(\zeta);\varsigma_n)}\,\sigma_1=\left(\begin{smallmatrix} \overline{d_n} & \overline{c_n}\\ \overline{b_n} & \overline{a_n}\end{smallmatrix}\right)$; thus,

$$\widehat{\chi}(\zeta)=\left(\mathrm{I}+\tfrac{1}{\zeta}\widehat{\Delta}_o\right)\widehat{\mathcal{P}}(\zeta)\begin{pmatrix} 1+\frac{a_n}{\zeta-\varsigma_n}+\sum_{\substack{k=1\\k\neq n}}^{m}\frac{a_k}{\zeta-\varsigma_k}+\sum_{k=1}^{m}\frac{\overline{d_k}}{\zeta-\overline{\varsigma_k}} & \frac{b_n}{\zeta-\varsigma_n}+\sum_{\substack{k=1\\k\neq n}}^{m}\frac{b_k}{\zeta-\varsigma_k}+\sum_{k=1}^{m}\frac{\overline{c_k}}{\zeta-\overline{\varsigma_k}}\\ \frac{c_n}{\zeta-\varsigma_n}+\sum_{\substack{k=1\\k\neq n}}^{m}\frac{c_k}{\zeta-\varsigma_k}+\sum_{k=1}^{m}\frac{\overline{b_k}}{\zeta-\overline{\varsigma_k}} & 1+\frac{d_n}{\zeta-\varsigma_n}+\sum_{\substack{k=1\\k\neq n}}^{m}\frac{d_k}{\zeta-\varsigma_k}+\sum_{k=1}^{m}\frac{\overline{a_k}}{\zeta-\overline{\varsigma_k}}\end{pmatrix}$$
$$\times\begin{pmatrix}\chi_{11}^c(\zeta) & \chi_{12}^c(\zeta)\\ \chi_{21}^c(\zeta) & \chi_{22}^c(\zeta)\end{pmatrix}. \tag{65}$$

As in the BC construction [41], one now Taylor expands $\chi^c(\zeta)$ about $\{\varsigma_n\}_{n=1}^m$: $\chi_{ij}^c(\zeta)=\chi_{ij}^c(\varsigma_n)+(\partial_\zeta\chi_{ij}^c(\varsigma_n))(\zeta-\varsigma_n)+\mathcal{O}((\zeta-\varsigma_n)^2)$, $i,j\in\{1,2\}$, where $\partial_\zeta\chi_{ij}^c(\varsigma_n)=\partial_\zeta\chi_{ij}^c(\zeta)|_{\zeta=\varsigma_n}$. Recalling from Lemma 3.5, *(iii)*, that $\widehat{\chi}(\zeta)$ satisfies the polar (residue) conditions $\mathrm{Res}(\widehat{\chi}(\zeta);\varsigma_n)=\lim_{\zeta\to\varsigma_n}\widehat{\chi}(\zeta)g_n^*\sigma_-$ and $\mathrm{Res}(\widehat{\chi}(\zeta);\overline{\varsigma_n})=\sigma_1\overline{\mathrm{Res}(\widehat{\chi}(\zeta);\varsigma_n)}\,\sigma_1$, $n\in\{1,2,\dots,m\}$, with g_n^* given in the Proposition, assembling the above, one shows that the only non-trivial conditions are

$$a_n\chi_{12}^c(\varsigma_n)+b_n\chi_{22}^c(\varsigma_n)=0,$$
$$c_n\chi_{12}^c(\varsigma_n)+d_n\chi_{22}^c(\varsigma_n)=0,$$
$$a_n\chi_{11}^c(\varsigma_n)+b_n\chi_{21}^c(\varsigma_n)=a_ng_n^*\partial_\zeta\chi_{12}^c(\varsigma_n)+\left(1+\sum_{\substack{k=1\\k\neq n}}^{m}\frac{a_k}{\varsigma_n-\varsigma_k}+\sum_{k=1}^{m}\frac{\overline{d_k}}{\varsigma_n-\overline{\varsigma_k}}\right)g_n^*\chi_{12}^c(\varsigma_n)$$
$$+b_ng_n^*\partial_\zeta\chi_{22}^c(\varsigma_n)+\left(\sum_{\substack{k=1\\k\neq n}}^{m}\frac{b_k}{\varsigma_n-\varsigma_k}+\sum_{k=1}^{m}\frac{\overline{c_k}}{\varsigma_n-\overline{\varsigma_k}}\right)g_n^*\chi_{22}^c(\varsigma_n)+\lim_{\zeta\to\varsigma_n}(\underbrace{a_n\chi_{12}^c(\varsigma_n)+b_n\chi_{22}^c(\varsigma_n)}_{0})\frac{g_n^*}{\zeta-\varsigma_n},$$
$$c_n\chi_{11}^c(\varsigma_n)+d_n\chi_{21}^c(\varsigma_n)=c_ng_n^*\partial_\zeta\chi_{12}^c(\varsigma_n)+\left(\sum_{\substack{k=1\\k\neq n}}^{m}\frac{c_k}{\varsigma_n-\varsigma_k}+\sum_{k=1}^{m}\frac{\overline{b_k}}{\varsigma_n-\overline{\varsigma_k}}\right)g_n^*\chi_{12}^c(\varsigma_n)$$
$$+d_ng_n^*\partial_\zeta\chi_{22}^c(\varsigma_n)+\left(1+\sum_{\substack{k=1\\k\neq n}}^{m}\frac{d_k}{\varsigma_n-\varsigma_k}+\sum_{k=1}^{m}\frac{\overline{a_k}}{\varsigma_n-\overline{\varsigma_k}}\right)g_n^*\chi_{22}^c(\varsigma_n)+\lim_{\zeta\to\varsigma_n}(\underbrace{c_n\chi_{12}^c(\varsigma_n)+d_n\chi_{22}^c(\varsigma_n)}_{0})\frac{g_n^*}{\zeta-\varsigma_n}.$$

From the first two equations of the above system, one gets that $b_n=-a_n\chi_{12}^c(\varsigma_n)/\chi_{22}^c(\varsigma_n)$ and $d_n=-c_n\chi_{12}^c(\varsigma_n)/\chi_{22}^c(\varsigma_n)$ (whence, $\det\left(\begin{smallmatrix} a_n & b_n\\ c_n & d_n\end{smallmatrix}\right)=0$): using the latter (two) relations, it follows from the last two equations of the above system that, for $n\in\{1,2,\dots,m\}$,

$$a_n\mathcal{A}_n=\sum_{\substack{k=1\\k\neq n}}^{m}\frac{a_kg_n^*\mathcal{B}_{nk}}{\varsigma_n-\varsigma_k}+\sum_{k=1}^{m}\frac{\overline{c_k}\,g_n^*\mathcal{D}_{nk}}{\varsigma_n-\overline{\varsigma_k}}+g_n^*\chi_{12}^c(\varsigma_n),\qquad \overline{c_n\mathcal{A}_n}=\sum_{\substack{k=1\\k\neq n}}^{m}\frac{\overline{c_kg_n^*\mathcal{B}_{nk}}}{\overline{\varsigma_n}-\overline{\varsigma_k}}+\sum_{k=1}^{m}\frac{a_k\overline{g_n^*\mathcal{D}_{nk}}}{\overline{\varsigma_n}-\varsigma_k}+\overline{g_n^*\chi_{22}^c(\varsigma_n)},$$

where

$$\mathcal{A}_n=\frac{\det(\chi^c(\varsigma_n))+g_n^*\mathrm{W}(\chi_{12}^c(\varsigma_n),\chi_{22}^c(\varsigma_n))}{\chi_{22}^c(\varsigma_n)},\qquad \mathcal{B}_{nk}=\frac{\chi_{12}^c(\varsigma_n)\chi_{22}^c(\varsigma_k)-\chi_{12}^c(\varsigma_k)\chi_{22}^c(\varsigma_n)}{\chi_{22}^c(\varsigma_k)},$$
$$\mathcal{D}_{nk}=\frac{\chi_{22}^c(\varsigma_n)\overline{\chi_{22}^c(\varsigma_k)}-\chi_{12}^c(\varsigma_n)\overline{\chi_{12}^c(\varsigma_k)}}{\overline{\chi_{22}^c(\varsigma_k)}};$$

thus, the (rank $2m$) linear inhomogeneous algebraic system for $\{a_n,\overline{c_n}\}_{n=1}^m$ stated in the Proposition. The non-degeneracy of the $(2m\times 2m)$ coefficient matrix is a consequence of the asymptotic solvability of the original RHP formulated in Lemma 2.1.2 [38] (see, also, Eq. (66) below). □

Proposition 4.2. *As* $t\to+\infty$ *and* $x\to-\infty$ *such that* $z_o:=x/t<-2$ *and* $(x,t)\in\urcorner_m$, $m\in\{1,2,\dots,N\}$, *for* $n\in\{1,2,\dots,m-1\}$,

$$a_n=\mathcal{O}\left(\mathrm{e}^{-\beth^+t}\right),\qquad b_n=\mathcal{O}\left(t^{-1/2}(z_o^2+32)^{-1/4}\mathrm{e}^{-\beth^+t}\right),$$
$$c_n=\mathcal{O}\left(\mathrm{e}^{-\beth^+t}\right),\qquad d_n=\mathcal{O}\left(t^{-1/2}(z_o^2+32)^{-1/4}\mathrm{e}^{-\beth^+t}\right),$$

where $\gimel^{+} := 4 \min_{\substack{m\in\{1,2,\dots,N\} \\ n\in\{1,2,\dots,m-1\}}} \{\sin(\phi_n)|\cos(\phi_n)-\cos(\phi_m)|\}$ (>0), *and*

$$\begin{aligned}
a_m =&\ a_m^0+\tfrac{1}{\sqrt{t}}a_m^1+\mathcal{O}\Big(\tfrac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\tfrac{\ln t}{t}\Big) \\
=:&\ \tfrac{g_m^{\ast}\overline{g_m^{\ast}}\,(\varsigma_m-\overline{\varsigma_m})^{-1}}{(1+g_m^{\ast}\overline{g_m^{\ast}}\,(\varsigma_m-\overline{\varsigma_m})^{-2})}+\tfrac{1}{\sqrt{t}}\Big(\tfrac{g_m^{\ast}\overline{g_m^{\ast}}\,(\varsigma_m-\overline{\varsigma_m})^{-1}(g_m^{\ast}\partial_{\zeta}\widetilde{\chi}_{12}^c(\varsigma_m)+\overline{g_m^{\ast}\partial_{\zeta}\widetilde{\chi}_{12}^c(\varsigma_m)})}{(1+g_m^{\ast}\overline{g_m^{\ast}}\,(\varsigma_m-\overline{\varsigma_m})^{-2})^2}+\tfrac{g_m^{\ast}\widetilde{\chi}_{12}^c(\varsigma_m)}{(1+g_m^{\ast}\overline{g_m^{\ast}}\,(\varsigma_m-\overline{\varsigma_m})^{-2})}\Big) \\
&+\ \mathcal{O}\Big(\tfrac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\tfrac{\ln t}{t}\Big), \\
b_m =&\ \tfrac{1}{\sqrt{t}}b_m^1+\mathcal{O}\Big(\tfrac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\tfrac{\ln t}{t}\Big)=:-\tfrac{1}{\sqrt{t}}\tfrac{g_m^{\ast}\overline{g_m^{\ast}}\,(\varsigma_m-\overline{\varsigma_m})^{-1}\widetilde{\chi}_{12}^c(\varsigma_m)}{(1+g_m^{\ast}\overline{g_m^{\ast}}\,(\varsigma_m-\overline{\varsigma_m})^{-2})}+\mathcal{O}\Big(\tfrac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\tfrac{\ln t}{t}\Big), \\
c_m =&\ c_m^0+\tfrac{1}{\sqrt{t}}c_m^1+\mathcal{O}\Big(\tfrac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\tfrac{\ln t}{t}\Big) \\
=:&\ \tfrac{g_m^{\ast}}{(1+g_m^{\ast}\overline{g_m^{\ast}}\,(\varsigma_m-\overline{\varsigma_m})^{-2})}+\tfrac{1}{\sqrt{t}}\Big(\tfrac{g_m^{\ast}\overline{g_m^{\ast}}\,(\varsigma_m-\overline{\varsigma_m})^{-1}\overline{\widetilde{\chi}_{12}^c(\varsigma_m)}-g_m^{\ast}\overline{g_m^{\ast}\partial_{\zeta}\widetilde{\chi}_{12}^c(\varsigma_m)}}{(1+g_m^{\ast}\overline{g_m^{\ast}}\,(\varsigma_m-\overline{\varsigma_m})^{-2})}+\tfrac{g_m^{\ast}(g_m^{\ast}\partial_{\zeta}\widetilde{\chi}_{12}^c(\varsigma_m)+\overline{g_m^{\ast}\partial_{\zeta}\widetilde{\chi}_{12}^c(\varsigma_m)})}{(1+g_m^{\ast}\overline{g_m^{\ast}}\,(\varsigma_m-\overline{\varsigma_m})^{-2})^2}\Big) \\
&+\ \mathcal{O}\Big(\tfrac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\tfrac{\ln t}{t}\Big), \\
d_m =&\ \tfrac{1}{\sqrt{t}}d_m^1+\mathcal{O}\Big(\tfrac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\tfrac{\ln t}{t}\Big)=:-\tfrac{1}{\sqrt{t}}\tfrac{g_m^{\ast}\widetilde{\chi}_{12}^c\,(\varsigma_m)}{(1+g_m^{\ast}\overline{g_m^{\ast}}\,(\varsigma_m-\overline{\varsigma_m})^{-2})}+\mathcal{O}\Big(\tfrac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\tfrac{\ln t}{t}\Big),
\end{aligned}$$

where

$$\widetilde{\chi}_{12}^c(\zeta)=\frac{\sqrt{\nu(\lambda_1)}\,\mathrm{e}^{\frac{\mathrm{i}\Xi^{+}(0)}{2}}\lambda_1^{2\mathrm{i}\nu(\lambda_1)}}{\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\left(\frac{\lambda_1\mathrm{e}^{-\mathrm{i}(\Theta^{+}(z_o,t)+\frac{\pi}{4})}}{(\zeta-\lambda_1)}+\frac{\lambda_2\mathrm{e}^{\mathrm{i}(\Theta^{+}(z_o,t)+\frac{\pi}{4})}}{(\zeta-\lambda_2)}\right),$$

with $\nu(\cdot)$, λ_1, λ_2, λ_3, $\Xi^{+}(\cdot)$, *and* $\Theta^{+}(z_o,t)$ *specified in Lemma* 4.2, *and* $c^{\mathcal{S}}(z_o)=\frac{c^{\mathcal{S}}(\lambda_1)\underline{c}(\lambda_2,\lambda_3,\overline{\lambda_3})}{\sqrt{\lambda_1}\,(\lambda_1-\lambda_2)}+\frac{c^{\mathcal{S}}(\lambda_2)\underline{c}(\lambda_1,\lambda_3,\overline{\lambda_3})}{\sqrt{\lambda_2}\,(\lambda_1-\lambda_2)}$.

Proof. Noting that, as $t\to+\infty$ and $x\to-\infty$ such that $z_o<-2$ and $(x,t)\in\daleth_m$, $g_n^{\ast}\!\upharpoonright_{\daleth_m}=\mathcal{O}(1)$, $n=m$, and $g_n^{\ast}\!\upharpoonright_{\daleth_m}=\mathcal{O}(\exp(-4t\sin(\phi_n)|\cos(\phi_n)-\cos(\phi_m)|))$, $n\in\{1,2,\dots,m-1\}$, one deduces from Proposition 4.1 that $\{a_n,\overline{c_n}\}_{n=1}^{m}$ solve

$$\left[\begin{array}{cc}
\boxed{\begin{array}{cccc}
\mathcal{A}_1 & o(1) & \cdots & o(1) \\
o(1) & \mathcal{A}_2 & \ddots & \vdots \\
\vdots & \ddots & \ddots & o(1) \\
-\frac{g_m^{\ast}\mathcal{B}_{m1}}{\varsigma_m-\varsigma_1} & \cdots & -\frac{g_m^{\ast}\mathcal{B}_{mm-1}}{\varsigma_m-\varsigma_{m-1}} & \mathcal{A}_m
\end{array}} &
\boxed{\begin{array}{cccc}
o(1) & \cdots & \cdots & o(1) \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
-\frac{g_m^{\ast}\mathcal{D}_{m1}}{\varsigma_m-\overline{\varsigma_1}} & \cdots & \cdots & -\frac{g_m^{\ast}\mathcal{D}_{mm}}{\varsigma_m-\overline{\varsigma_m}}
\end{array}} \\
\boxed{\begin{array}{cccc}
o(1) & \cdots & \cdots & o(1) \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
-\frac{\overline{g_m^{\ast}\mathcal{D}_{m1}}}{\overline{\varsigma_m}-\varsigma_1} & \cdots & \cdots & -\frac{\overline{g_m^{\ast}\mathcal{D}_{mm}}}{\overline{\varsigma_m}-\varsigma_m}
\end{array}} &
\boxed{\begin{array}{cccc}
\overline{\mathcal{A}_1} & o(1) & \cdots & o(1) \\
\vdots & \overline{\mathcal{A}_2} & \ddots & \vdots \\
\vdots & \ddots & \ddots & o(1) \\
-\frac{\overline{g_m^{\ast}\mathcal{B}_{m1}}}{\overline{\varsigma_m}-\overline{\varsigma_1}} & \cdots & -\frac{\overline{g_m^{\ast}\mathcal{B}_{mm-1}}}{\overline{\varsigma_m}-\overline{\varsigma_{m-1}}} & \overline{\mathcal{A}_m}
\end{array}}
\end{array}\right]
\left[\begin{array}{c}
a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_m \\ \overline{c_1} \\ \overline{c_2} \\ \vdots \\ \vdots \\ \overline{c_m}
\end{array}\right]$$

$$=[\underbrace{o(1),\dots,o(1),g_m^{\ast}\chi_{12}^c(\varsigma_m)}_{m},\underbrace{o(1),\dots,o(1),\overline{g_m^{\ast}\chi_{22}^c(\varsigma_m)}}_{m}]^{\mathrm{T}},$$

where $^{\mathrm{T}}$ denotes transposition, $\mathcal{A}_n$, $\mathcal{B}_{nk}$, and $\mathcal{D}_{nk}$ are given in the proof of Proposition 4.1, $o(1):=\mathcal{O}(\exp(-4t\min_{\substack{m\in\{1,2,\dots,N\} \\ n\in\{1,2,\dots,m-1\}}}\{\sin(\phi_n)|\cos(\phi_n)-\cos(\phi_m)|\}))$, and $\chi_{12}^c(\cdot)$ and $\chi_{22}^c(\cdot)$ are given in Lemma 4.2. Solving the above system for $\{a_n,\overline{c_n}\}_{n=1}^{m}$ via the Cauchy-Binet formula, or Cramer's Rule, recalling the expressions for $\chi_{ij}^c(\zeta)$, $i,j\in\{1,2\}$, given in Lemma 4.2, setting $\widetilde{\chi}_{12}^c(\zeta)$ and $c^{\mathcal{S}}(z_o)$ as in the Proposition, and recalling from Proposition 4.1 that $b_n=-a_n\chi_{12}^c(\varsigma_n)/\chi_{22}^c(\varsigma_n)$ and $d_n=-c_n\chi_{12}^c(\varsigma_n)/\chi_{22}^c(\varsigma_n)$, one gets the estimates for $\{a_n,b_n,c_n,d_n\}_{n=1}^{m-1}$ and the explicit—asymptotic expansion—formulae for $\{a_m,b_m,c_m,d_m\}$ stated in the Proposition. Furthermore, setting $\mathcal{Y}:=\begin{pmatrix}\boxed{\widehat{\mathbb{A}}} & \boxed{\widehat{\mathbb{B}}} \\ \boxed{\overline{\widehat{\mathbb{B}}}} & \boxed{\overline{\widehat{\mathbb{A}}}}\end{pmatrix}$, with $\widehat{\mathbb{A}}$ and $\widehat{\mathbb{B}}$ defined in Proposition 4.1, from the asymptotic estimates above for $\{a_n,b_n,c_n,d_n\}_{n=1}^{m}$, and recalling that, as a consequence of the asymptotic solvability of the original RHP formulated in Lemma 2.1.2,

$\det(\mathcal{Y})\not\equiv 0$, an application of Hadamard's Inequality ($|\det(\mathcal{Y})|^2 \leqslant \prod_{j=1}^{2m}\sum_{i=1}^{2m}|\mathcal{Y}_{ij}|^2$, where $\mathcal{Y}_{ij}$ denotes the $(i\,j)$-element of $\mathcal{Y}$) shows that

$$0<|\det(\mathcal{Y})|^2 \leqslant \prod_{j=1}^{m}\Big(1+\tfrac{\sin^2(\phi_m)|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)}{\sin^2(\frac{1}{2}(\phi_m+\phi_j))}\mathrm{e}^{2\phi(x,t)}\Big)^2+\mathcal{O}\Big(\tfrac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\tfrac{\ln t}{t}\Big), \tag{66}$$

where

$$\phi(x,t):=-2\sin(\phi_m)(x+2t\cos\phi_m), \tag{67}$$

$$P(\phi_m,\phi_k):=\left(\prod_{k=1}^{m-1}\frac{\sin(\frac{1}{2}(\phi_m+\phi_k))}{\sin(\frac{1}{2}(\phi_m-\phi_k))}\right)\left(\prod_{k=m+1}^{N}\frac{\sin(\frac{1}{2}(\phi_m+\phi_k))}{\sin(\frac{1}{2}(\phi_m-\phi_k))}\right)^{-1}, \tag{68}$$

$$Q(\phi_m):=\exp\left(\left(\int_0^{\lambda_2}+\int_{\lambda_1}^{+\infty}-\int_{-\infty}^{0}-\int_{\lambda_2}^{\lambda_1}\right)\frac{\sin(\phi_m)\ln(1-|r(\mu)|^2)}{(\mu^2-2\mu\cos(\phi_m)+1)}\frac{\mathrm{d}\mu}{2\pi}\right), \tag{69}$$

and $c^{\mathcal{S}}(z_o)$ is given in the Proposition. □

The following Lemma is proved via the higher-order generalisation [54] of the Deift-Zhou (DZ) non-linear steepest descent method [55] (see, also, [56]), but its proof is far beyond the scope of the present work (it shall be presented elsewhere).

Remark 4.1. Even though in Lemma 4.3 below, in the *sensus strictu* of asymptotic analysis, the exponentially small terms should be neglected, and thus not written out explicitly, in lieu of the $t^{-p/2}(\ln t)^q$ corrections, $p\geqslant 1$, $q\in\{0,1,\ldots,p-1\}$, they are written there, and there only (see, also, Appendix A, Lemma A.1.7), in order to bring to the reader's attention the fact that there are additional, albeit exponentially small, terms that are due to the remaining solitons: thereafter, exponentially small terms are neglected.

Lemma 4.3. *As* $t\to+\infty$ *and* $x\to-\infty$ *such that* $z_o:=x/t<-2$ *and* $(x,t)\in\urcorner_m$, $m\in\{1,2,\ldots,N\}$,

$$\widehat{\mathcal{P}}(\zeta)=\begin{pmatrix}\frac{\zeta+\widehat{a}_1^+}{\zeta+\widehat{a}_2^+} & \frac{\widehat{a}_3^+}{\zeta+\widehat{a}_4^+}\\ \frac{\overline{\widehat{a}_3^+}}{\zeta+\widehat{a}_4^+} & \frac{\zeta+\overline{\widehat{a}_1^+}}{\zeta+\overline{\widehat{a}_2^+}}\end{pmatrix},$$

where

$$\widehat{a}_1^+=\overline{\widehat{a}_2^+}=1+\sum_{p=1}^{\infty}\sum_{q=0}^{p-1}\frac{\widehat{a}_{pq}^1(z_o)(\ln t)^q}{t^{p/2}}+\mathcal{O}\left(\mathrm{e}^{-4t\min_{\substack{m\in\{1,2,\ldots,N\}\\ n\in\{1,2,\ldots,m-1\}}}\{\sin(\phi_n)|\cos(\phi_n)-\cos(\phi_m)|\}}\right),$$

$$\widehat{a}_3^+=\sum_{p=1}^{\infty}\sum_{q=0}^{p-1}\frac{\widehat{a}_{pq}^3(z_o)(\ln t)^q}{t^{p/2}}+\mathcal{O}\left(\mathrm{e}^{-4t\min_{\substack{m\in\{1,2,\ldots,N\}\\ n\in\{1,2,\ldots,m-1\}}}\{\sin(\phi_n)|\cos(\phi_n)-\cos(\phi_m)|\}}\right),$$

$$\widehat{a}_4^+=1+\sum_{p=1}^{\infty}\sum_{q=0}^{p-1}\frac{\widehat{a}_{pq}^4(z_o)(\ln t)^q}{t^{p/2}}+\mathcal{O}\left(\mathrm{e}^{-4t\min_{\substack{m\in\{1,2,\ldots,N\}\\ n\in\{1,2,\ldots,m-1\}}}\{\sin(\phi_n)|\cos(\phi_n)-\cos(\phi_m)|\}}\right),$$

$\widehat{a}_{pq}^k(z_o)\in c^{\mathcal{S}}(z_o)$, $k\in\{1,3,4\}$, *and* $\widehat{\mathcal{P}}(\zeta)=\sigma_1\overline{\widehat{\mathcal{P}}(\overline{\zeta})}\,\sigma_1$.

Remark 4.2. Even though Lemma 4.3 is not proven in this paper, it will be shown that (see the proof of Proposition 4.6 below), up to the leading-order terms retained in this work, namely, terms that are $\mathcal{O}(\frac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\frac{\ln t}{t})$, $\widehat{a}_{10}^3(z_o)=0$; thus, actually, $\widehat{a}_3^+=\sum_{p=2}^{\infty}\sum_{q=0}^{p-1}\frac{\widehat{a}_{pq}^3(z_o)(\ln t)^q}{t^{p/2}}+\mathcal{O}(\exp(-4t\min_{\substack{m\in\{1,2,\ldots,N\}\\ n\in\{1,2,\ldots,m-1\}}}\{\sin(\phi_n)|\cos(\phi_n)-\cos(\phi_m)|\}))=\mathcal{O}(\frac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\frac{\ln t}{t})$. Furthermore, to $\mathcal{O}(\frac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\frac{\ln t}{t})$, the asymptotic expansion for $\widehat{a}_4^+$ plays, in fact, no role in the final formulae of this paper. As a possible prelude to a motivation of why $\widehat{a}_i^+$, $i\in\{1,2,3,4\}$, have, modulo exponentially small terms, the asymptotic expansions stated in Lemma 4.3, one can apply the higher-order generalisation of the DZ method [54] to the proof of Lemma 6.1 in [38] to show that $\chi_{ij}^c(\zeta)$, $i,j\in\{1,2\}$, have the asymptotic expansion $\chi_{ij}^c(\zeta)=\delta_{ij}+\sum_{p=1}^{\infty}\sum_{q=0}^{p-1}\frac{(\chi_{ij}^c(z_o))_{pq}(f_{ij}(\zeta))_{pq}(\ln t)^q}{t^{p/2}}$, where δ_{ij} is the Kronecker delta, $(\chi_{11}^c(\cdot))_{10}=(\chi_{22}^c(\cdot))_{10}=0$, and $\|(f_{ij}(\cdot))_{pq}\|_{\mathcal{L}^\infty(\mathbb{C}\setminus\cup_{z\in\{\lambda_1,\lambda_2\}}\mathbb{U}(z;\varepsilon))}<\infty$; however, as stated heretofore, these details are omitted in this paper (it is the author's conjecture that $\widehat{a}_3^+=\widehat{a}_4^+=0$, namely, $\widehat{\mathcal{P}}(\zeta)$ is diagonal).

Proposition 4.3. *Set* $\widehat{a}_{10}^{1}(z_o)=:\widehat{a}_1$, $\widehat{a}_{10}^{2}(z_o)=:\widehat{a}_2$, $\widehat{a}_{10}^{3}(z_o)=:\widehat{a}_3$, *and* $\widehat{a}_{10}^{4}(z_o)=:\widehat{a}_4$. *Then as* $t\to+\infty$ *and* $x\to-\infty$ *such that* $z_o:=x/t<-2$ *and* $(x,t)\in\urcorner_m$, $m\in\{1,2,\dots,N\}$,

$$
\begin{aligned}
(\widehat{\Delta}_o)_{11}=&-\frac{\overline{c_m^0}}{\overline{\varsigma_m}}\mathrm{i}\delta^{-1}(0)\mathrm{e}^{2\mathrm{i}\sum_{k=m+1}^{N}\phi_k}+\frac{\mathrm{i}\delta^{-1}(0)\mathrm{e}^{2\mathrm{i}\sum_{k=m+1}^{N}\phi_k}}{\sqrt{t}}\left(-(\widehat{a}_1-\widehat{a}_2)\frac{\overline{c_m^0}}{\overline{\varsigma_m}}-\left(\frac{b_m^1}{\varsigma_m}+\frac{\overline{c_m^1}}{\overline{\varsigma_m}}\right)\right.\\
&\left.+\,\widehat{a}_3\left(1-\frac{\overline{a_m^0}}{\overline{\varsigma_m}}\right)-\left(1-\frac{a_m^0}{\varsigma_m}\right)\frac{2\delta(0)\sqrt{\nu(\lambda_1)}\,\cos(\Theta^+(z_o,t)+\frac{\pi}{4})}{\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\right)+\mathcal{O}\left(\frac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\frac{\ln t}{t}\right),\\
(\widehat{\Delta}_o)_{12}=&-\left(1-\frac{a_m^0}{\varsigma_m}\right)\mathrm{i}\delta(0)\mathrm{e}^{-2\mathrm{i}\sum_{k=m+1}^{N}\phi_k}+\frac{\mathrm{i}\delta(0)\mathrm{e}^{-2\mathrm{i}\sum_{k=m+1}^{N}\phi_k}}{\sqrt{t}}\left(-(\widehat{a}_1-\widehat{a}_2)\left(1-\frac{a_m^0}{\varsigma_m}\right)+\left(\frac{a_m^1}{\varsigma_m}+\frac{\overline{d_m^1}}{\overline{\varsigma_m}}\right)\right.\\
&\left.+\,\widehat{a}_3\frac{c_m^0}{\varsigma_m}-\frac{\overline{c_m^0}}{\overline{\varsigma_m}}\frac{2\delta^{-1}(0)\sqrt{\nu(\lambda_1)}\,\cos(\Theta^+(z_o,t)+\frac{\pi}{4})}{\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\right)+\mathcal{O}\left(\frac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\frac{\ln t}{t}\right),\\
(\widehat{\Delta}_o)_{21}=&\left(1-\frac{\overline{a_m^0}}{\overline{\varsigma_m}}\right)\mathrm{i}\delta^{-1}(0)\mathrm{e}^{2\mathrm{i}\sum_{k=m+1}^{N}\phi_k}+\frac{\mathrm{i}\delta^{-1}(0)\mathrm{e}^{2\mathrm{i}\sum_{k=m+1}^{N}\phi_k}}{\sqrt{t}}\left((\overline{\widehat{a}_1}-\overline{\widehat{a}_2})\left(1-\frac{\overline{a_m^0}}{\overline{\varsigma_m}}\right)-\left(\frac{\overline{a_m^1}}{\overline{\varsigma_m}}+\frac{d_m^1}{\varsigma_m}\right)\right.\\
&\left.-\,\overline{\widehat{a}_3}\frac{\overline{c_m^0}}{\overline{\varsigma_m}}+\frac{c_m^0}{\varsigma_m}\frac{2\delta(0)\sqrt{\nu(\lambda_1)}\,\cos(\Theta^+(z_o,t)+\frac{\pi}{4})}{\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\right)+\mathcal{O}\left(\frac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\frac{\ln t}{t}\right),\\
(\widehat{\Delta}_o)_{22}=&\frac{c_m^0}{\varsigma_m}\mathrm{i}\delta(0)\mathrm{e}^{-2\mathrm{i}\sum_{k=m+1}^{N}\phi_k}+\frac{\mathrm{i}\delta(0)\mathrm{e}^{-2\mathrm{i}\sum_{k=m+1}^{N}\phi_k}}{\sqrt{t}}\left((\overline{\widehat{a}_1}-\overline{\widehat{a}_2})\frac{c_m^0}{\varsigma_m}+\left(\frac{\overline{b_m^1}}{\overline{\varsigma_m}}+\frac{c_m^1}{\varsigma_m}\right)\right.\\
&\left.-\,\overline{\widehat{a}_3}\left(1-\frac{a_m^0}{\varsigma_m}\right)+\left(1-\frac{\overline{a_m^0}}{\overline{\varsigma_m}}\right)\frac{2\delta^{-1}(0)\sqrt{\nu(\lambda_1)}\,\cos(\Theta^+(z_o,t)+\frac{\pi}{4})}{\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\right)+\mathcal{O}\left(\frac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\frac{\ln t}{t}\right).
\end{aligned}
$$

Proof. Recall from Lemma 4.1 that $\widehat{\Delta}_o=\widehat{\mathcal{P}}(0)\widehat{m}_d(0)\chi^c(0)(\delta(0))^{\sigma_3}\left(\prod_{k=m+1}^{N}(d_k^+(0))^{\sigma_3}\right)\sigma_2$. Collect, now, the following facts: (1) from Lemma 4.3, $\widehat{\mathcal{P}}(0)=\begin{pmatrix}\widehat{a}_1^+/\widehat{a}_2^+ & \widehat{a}_3^+/\widehat{a}_4^+\\ \widehat{a}_3^+/\widehat{a}_4^+ & \widehat{a}_1^+/\widehat{a}_2^+\end{pmatrix}$; (2) from the expression for $\widehat{m}_d(\zeta)$ given in Lemma 4.1, the definition (cf. Proposition 4.1) $\mathrm{Res}(\widehat{\chi}(\zeta);\varsigma_n)=\left(\begin{smallmatrix}a_n & b_n\\ c_n & d_n\end{smallmatrix}\right)$, $n\in\{1,2,\dots,m\}$, and the asymptotics for $\{a_n,b_n,c_n,d_n\}_{n=1}^{m}$ given in Proposition 4.2, one shows that

$$
\widehat{m}_d(0)=\begin{pmatrix}1-\frac{a_m}{\varsigma_m}-\frac{\overline{d_m}}{\overline{\varsigma_m}} & -\frac{b_m}{\varsigma_m}-\frac{\overline{c_m}}{\overline{\varsigma_m}}\\ -\frac{\overline{b_m}}{\overline{\varsigma_m}}-\frac{c_m}{\varsigma_m} & 1-\frac{\overline{a_m}}{\overline{\varsigma_m}}-\frac{d_m}{\varsigma_m}\end{pmatrix}+\mathcal{O}_{2\times2}\left(\mathrm{e}^{-4t\min_{\substack{m\in\{1,2,\dots,N\}\\ n\in\{1,2,\dots,m-1\}}}\{\sin(\phi_n)|\cos(\phi_n)-\cos(\phi_m)|\}}\right),
$$

where $\mathcal{O}_{2\times2}(\blacklozenge)$ denotes a 2×2 matrix each of whose entries are $\mathcal{O}(\blacklozenge)$; (3) from Lemma 4.2 and the formula for $\widetilde{\chi}_{12}^c(\zeta)$ $(=\overline{\widetilde{\chi}_{21}^c(\overline{\zeta})})$ given in Proposition 4.2, one shows that $\chi^c(0)=\begin{pmatrix}1 & \frac{1}{\sqrt{t}}\widetilde{\chi}_{12}^c(0)\\ \frac{1}{\sqrt{t}}\widetilde{\chi}_{21}^c(0) & 1\end{pmatrix}+\mathcal{O}_{2\times2}(\frac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\frac{\ln t}{t})$; and (4) $(\delta(0))^{\sigma_3}\left(\prod_{k=m+1}^{N}(d_k^+(0))^{\sigma_3}\right)\sigma_2=\mathrm{i}\delta^{-1}(0)\left(\prod_{k=m+1}^{N}(d_k^+(0))^{-1}\right)\sigma_--\mathrm{i}\delta(0)\cdot\left(\prod_{k=m+1}^{N}d_k^+(0)\right)\sigma_+$. Using the results of (1)–(4), and recalling the expression for $\widehat{\Delta}_o$ given above, one arrives at

$$
\begin{aligned}
(\widehat{\Delta}_o)_{11}=&\frac{\widehat{a}_1^+}{\widehat{a}_2^+}\mathrm{i}\delta^{-1}(0)\Big(\frac{\widetilde{\chi}_{12}^c(0)}{\sqrt{t}}(1-\tfrac{a_m}{\varsigma_m}-\tfrac{\overline{d_m}}{\overline{\varsigma_m}})-(\tfrac{b_m}{\varsigma_m}+\tfrac{\overline{c_m}}{\overline{\varsigma_m}})\Big)\prod_{k=m+1}^{N}(d_k^+(0))^{-1}\\
&+\frac{\widehat{a}_3^+}{\widehat{a}_4^+}\mathrm{i}\delta^{-1}(0)\Big((1-\tfrac{\overline{a_m}}{\overline{\varsigma_m}}-\tfrac{d_m}{\varsigma_m})-\frac{\widetilde{\chi}_{12}^c(0)}{\sqrt{t}}(\tfrac{\overline{b_m}}{\overline{\varsigma_m}}+\tfrac{c_m}{\varsigma_m})\Big)\prod_{k=m+1}^{N}(d_k^+(0))^{-1},\\
(\widehat{\Delta}_o)_{12}=&\frac{\widehat{a}_1^+}{\widehat{a}_2^+}\mathrm{i}\delta(0)\Big(-(1-\tfrac{a_m}{\varsigma_m}-\tfrac{\overline{d_m}}{\overline{\varsigma_m}})+\frac{\widetilde{\chi}_{21}^c(0)}{\sqrt{t}}(\tfrac{b_m}{\varsigma_m}+\tfrac{\overline{c_m}}{\overline{\varsigma_m}})\Big)\prod_{k=m+1}^{N}d_k^+(0)\\
&+\frac{\widehat{a}_3^+}{\widehat{a}_4^+}\mathrm{i}\delta(0)\Big(-\frac{\widetilde{\chi}_{21}^c(0)}{\sqrt{t}}(1-\tfrac{\overline{a_m}}{\overline{\varsigma_m}}-\tfrac{d_m}{\varsigma_m})+(\tfrac{\overline{b_m}}{\overline{\varsigma_m}}+\tfrac{c_m}{\varsigma_m})\Big)\prod_{k=m+1}^{N}d_k^+(0),
\end{aligned}
$$

$$(\widehat{\Delta}_o)_{21} = \frac{\overline{\widehat{a}_1^+}}{\widehat{a}_2^+}\mathrm{i}\delta^{-1}(0)\Big((1-\tfrac{\overline{a_m}}{\varsigma_m}-\tfrac{d_m}{\varsigma_m})-\tfrac{\widetilde{\chi}_{12}^c(0)}{\sqrt{t}}(\tfrac{\overline{b_m}}{\overline{\varsigma_m}}+\tfrac{c_m}{\varsigma_m})\Big)\prod_{k=m+1}^{N}(d_k^+(0))^{-1}$$
$$+\frac{\overline{\widehat{a}_3^+}}{\widehat{a}_4^+}\mathrm{i}\delta^{-1}(0)\Big(\tfrac{\widetilde{\chi}_{12}^c(0)}{\sqrt{t}}(1-\tfrac{a_m}{\varsigma_m}-\tfrac{\overline{d_m}}{\overline{\varsigma_m}})-(\tfrac{b_m}{\varsigma_m}+\tfrac{\overline{c_m}}{\overline{\varsigma_m}})\Big)\prod_{k=m+1}^{N}(d_k^+(0))^{-1},$$
$$(\widehat{\Delta}_o)_{22} = \frac{\overline{\widehat{a}_1^+}}{\widehat{a}_2^+}\mathrm{i}\delta(0)\Big(-\tfrac{\widetilde{\chi}_{21}^c(0)}{\sqrt{t}}(1-\tfrac{\overline{a_m}}{\overline{\varsigma_m}}-\tfrac{d_m}{\varsigma_m})+(\tfrac{\overline{b_m}}{\overline{\varsigma_m}}+\tfrac{c_m}{\varsigma_m})\Big)\prod_{k=m+1}^{N}d_k^+(0)$$
$$+\frac{\overline{\widehat{a}_3^+}}{\widehat{a}_4^+}\mathrm{i}\delta(0)\Big(-(1-\tfrac{a_m}{\varsigma_m}-\tfrac{\overline{d_m}}{\overline{\varsigma_m}})+\tfrac{\widetilde{\chi}_{21}^c(0)}{\sqrt{t}}(\tfrac{b_m}{\varsigma_m}+\tfrac{\overline{c_m}}{\overline{\varsigma_m}})\Big)\prod_{k=m+1}^{N}d_k^+(0).$$

Using the asymptotic expansions for $\{a_m,b_m,c_m,d_m\}$ (respectively, $\{\widehat{a}_i^+\}_{i=1}^4$) given in Proposition 4.2 (respectively, Lemma 4.3), one arrives at the leading-order results stated in the Proposition. □

Remark 4.3. In Propositions 4.4 and 4.6 below, one should keep, everywhere, the upper (respectively, lower) signs for $\theta_{\gamma_m}=+\pi/2$ (respectively, $\theta_{\gamma_m}=-\pi/2$).

Proposition 4.4. *Let $\phi(x,t)$, $P(\phi_m,\phi_k)$, and $Q(\phi_m)$ be defined by Eqs.* (67)*,* (68)*, and* (69)*, respectively. Then, for $\theta_{\gamma_m}=\pm\pi/2$, as $t\to+\infty$ and $x\to-\infty$ such that $z_o:=x/t<-2$ and $(x,t)\in\urcorner_m$, $m\in\{1,2,\dots,N\}$,*

$$a_m^0 = -\frac{2\mathrm{i}\sin(\phi_m)|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})},$$
$$c_m^0 = \mp\frac{2\sin(\phi_m)|\gamma_m|\delta^{-1}(0)\mathrm{e}^{\mathrm{i}(\phi_m+s^+)+\phi(x,t)}P(\phi_m,\phi_k)Q(\phi_m)}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})},$$
$$\begin{aligned}
a_m^1 = &\mp\frac{16\mathrm{i}\lambda_1^2\sin^2(\phi_m)|\gamma_m|^3\sqrt{\nu(\lambda_1)}\,P^3(\phi_m,\phi_k)Q^3(\phi_m)\cos(s^+)\mathrm{e}^{3\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})^2(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)^2\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\\
&\times\big(((\lambda_1+\lambda_2)\cos(\phi_m)-2)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})+(\lambda_1-\lambda_2)\sin(\phi_m)\sin(\Theta^+(z_o,t)+\tfrac{\pi}{4})\big)\\
&\mp\frac{2\lambda_1\sin(\phi_m)|\gamma_m|\sqrt{\nu(\lambda_1)}\,P(\phi_m,\phi_k)Q(\phi_m)\mathrm{e}^{\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\\
&\times\big(2\cos(s^+)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})-(\lambda_1+\lambda_2)\cos(\phi_m+s^+)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})\\
&-(\lambda_1-\lambda_2)\sin(\phi_m+s^+)\sin(\Theta^+(z_o,t)+\tfrac{\pi}{4})+2\mathrm{i}\sin(s^+)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})\\
&-\mathrm{i}(\lambda_1+\lambda_2)\sin(\phi_m+s^+)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})+\mathrm{i}(\lambda_1-\lambda_2)\cos(\phi_m+s^+)\sin(\Theta^+(z_o,t)+\tfrac{\pi}{4})\big),
\end{aligned}$$
$$\begin{aligned}
b_m^1 = &\frac{2\mathrm{i}\lambda_1\sin(\phi_m)|\gamma_m|^2\sqrt{\nu(\lambda_1)}\,\delta(0)\mathrm{e}^{-\mathrm{i}(\phi_m+s^+)+2\phi(x,t)}P^2(\phi_m,\phi_k)Q^2(\phi_m)}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\\
&\times\big(2\cos(s^+)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})-(\lambda_1+\lambda_2)\cos(\phi_m+s^+)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})\\
&-(\lambda_1-\lambda_2)\sin(\phi_m+s^+)\sin(\Theta^+(z_o,t)+\tfrac{\pi}{4})+2\mathrm{i}\sin(s^+)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})\\
&-\mathrm{i}(\lambda_1+\lambda_2)\sin(\phi_m+s^+)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})+\mathrm{i}(\lambda_1-\lambda_2)\cos(\phi_m+s^+)\sin(\Theta^+(z_o,t)+\tfrac{\pi}{4})\big),
\end{aligned}$$
$$\begin{aligned}
c_m^1 = &-\frac{16\lambda_1^2\sin^2(\phi_m)|\gamma_m|^2\sqrt{\nu(\lambda_1)}\,\delta^{-1}(0)\mathrm{e}^{\mathrm{i}(\phi_m+s^+)+2\phi(x,t)}P^2(\phi_m,\phi_k)Q^2(\phi_m)\cos(s^+)}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})^2(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)^2\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\\
&\times\big(((\lambda_1+\lambda_2)\cos(\phi_m)-2)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})+(\lambda_1-\lambda_2)\sin(\phi_m)\sin(\Theta^+(z_o,t)+\tfrac{\pi}{4})\big)\\
&-\frac{2\mathrm{i}\lambda_1\sin(\phi_m)|\gamma_m|^2\sqrt{\nu(\lambda_1)}\,\delta^{-1}(0)\mathrm{e}^{\mathrm{i}(\phi_m+s^+)+2\phi(x,t)}P^2(\phi_m,\phi_k)Q^2(\phi_m)}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\\
&\times\big(2\cos(s^+)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})-(\lambda_1+\lambda_2)\cos(\phi_m+s^+)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})\\
&-(\lambda_1-\lambda_2)\sin(\phi_m+s^+)\sin(\Theta^+(z_o,t)+\tfrac{\pi}{4})-2\mathrm{i}\sin(s^+)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})\\
&+\mathrm{i}(\lambda_1+\lambda_2)\sin(\phi_m+s^+)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})-\mathrm{i}(\lambda_1-\lambda_2)\cos(\phi_m+s^+)\sin(\Theta^+(z_o,t)+\tfrac{\pi}{4})\big)\\
&+\frac{8\lambda_1^2\sin^2(\phi_m)|\gamma_m|^2\sqrt{\nu(\lambda_1)}\,\delta^{-1}(0)\mathrm{e}^{\mathrm{i}(\phi_m+s^+)+2\phi(x,t)}P^2(\phi_m,\phi_k)Q^2(\phi_m)}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)^2\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}
\end{aligned}$$

$$\times\big(\big(((\lambda_1+\lambda_2)\cos(\phi_m)-2)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})+(\lambda_1-\lambda_2)\sin(\phi_m)\sin(\Theta^+(z_o,t)+\tfrac{\pi}{4})\big)\cos(s^+)$$
$$-\,\mathrm{i}\big(((\lambda_1+\lambda_2)\cos(\phi_m)-2)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})+(\lambda_1-\lambda_2)\sin(\phi_m)\sin(\Theta^+(z_o,t)+\tfrac{\pi}{4})\big)\sin(s^+)\big)\,,$$
$$d_m^1=\pm\,\frac{2\lambda_1\sin(\phi_m)|\gamma_m|\sqrt{\nu(\lambda_1)}\,P(\phi_m,\phi_k)Q(\phi_m)\mathrm{e}^{\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}$$
$$\times\big(2\cos(s^+)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})-(\lambda_1+\lambda_2)\cos(\phi_m+s^+)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})$$
$$-\,(\lambda_1-\lambda_2)\sin(\phi_m+s^+)\sin(\Theta^+(z_o,t)+\tfrac{\pi}{4})+2\mathrm{i}\sin(s^+)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})$$
$$-\,\mathrm{i}(\lambda_1+\lambda_2)\sin(\phi_m+s^+)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})+\mathrm{i}(\lambda_1-\lambda_2)\cos(\phi_m+s^+)\sin(\Theta^+(z_o,t)+\tfrac{\pi}{4})\big)\,,$$

where s^+ is given in Theorem 2.2.1, *Eq.* (11).

Proof. Recalling the definitions of $\{a_m^0,a_m^1,b_m^1,c_m^0,c_m^1,d_m^1\}$ given in Proposition 4.2, substituting into them the expressions for g_m^* and $\widetilde{\chi}_{12}^c(\zeta)$ given in Propositions 4.1 and 4.2, respectively, using standard trigonometric identities, and defining $\phi(x,t)$, $P(\phi_m,\phi_k)$, and $Q(\phi_m)$ as in Eqs. (67), (68), and (69), respectively, one obtains, after tedious, but otherwise straightforward calculations, the result stated in the Proposition. □

Proposition 4.5. *As $t\to+\infty$ and $x\to-\infty$ such that $z_o:=x/t<-2$ and $(x,t)\in\urcorner_m$, $m\in\{1,2,\dots,N\}$,*

$$u(x,t)=\mathrm{i}\left((\widehat{\Delta}_o)_{12}+\widehat{a}_3^++b_m+\overline{c_m}+\frac{\sqrt{\nu(\lambda_1)}\,\mathrm{e}^{\frac{\mathrm{i}\Xi^+(0)}{2}}\lambda_1^{2\mathrm{i}\nu(\lambda_1)}}{\sqrt{t(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\left(\lambda_1\mathrm{e}^{-\mathrm{i}(\Theta^+(z_o,t)+\frac{\pi}{4})}+\lambda_2\mathrm{e}^{\mathrm{i}(\Theta^+(z_o,t)+\frac{\pi}{4})}\right)\right)$$
$$+\,\mathcal{O}\left(\tfrac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\tfrac{\ln t}{t}\right),\tag{70}$$

$$\int_{+\infty}^{x}(|u(x',t)|^2-1)\,\mathrm{d}x'=-\,\mathrm{i}\left((\widehat{\Delta}_o)_{11}+\widehat{a}_1^+-\widehat{a}_2^++a_m+\overline{d_m}+2\mathrm{i}\sum_{k=m+1}^{N}\sin(\phi_k)\right.$$
$$\left.+\,\mathrm{i}\left(\int_{-\infty}^{0}+\int_{\lambda_2}^{\lambda_1}\right)\ln(1-|r(\mu)|^2)\,\tfrac{\mathrm{d}\mu}{2\pi}\right)+\mathcal{O}\left(\tfrac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\tfrac{\ln t}{t}\right),\tag{71}$$

$$\int_{-\infty}^{x}(|u(x',t)|^2-1)\,\mathrm{d}x'=\int_{+\infty}^{x}(|u(x',t)|^2-1)\,\mathrm{d}x'-2\sum_{n=1}^{N}\sin(\phi_n)-\int_{-\infty}^{+\infty}\ln(1-|r(\mu)|^2)\,\tfrac{\mathrm{d}\mu}{2\pi}.\tag{72}$$

Proof. Recall Eqs. (63), (64), and (65) for $u(x,t)$, $\int_{+\infty}^{x}(|u(x',t)|^2-1)\,\mathrm{d}x'$, and $\widehat{\chi}(\zeta)$, respectively. Using the result for $\widehat{\mathcal{P}}(\zeta)$ (respectively, $\chi_{ij}^c(\zeta)$, $i,j\in\{1,2\}$) stated in Lemma 4.3 (respectively, Lemma 4.2), noting that $(\delta(\zeta))^{\pm1}=_{\zeta\to\infty}1\pm\mathrm{i}\Big(\Big(\int_{-\infty}^{0}+\int_{\lambda_2}^{\lambda_1}\Big)\ln(1-|r(\mu)|^2)\frac{\mathrm{d}\mu}{2\pi}\Big)\zeta^{-1}+\mathcal{O}(\zeta^{-2})$ and $\prod_{k=m+1}^{N}(d_k^+(\zeta))^{\pm1}=_{\zeta\to\infty}1\pm\Big(\sum_{k=m+1}^{N}(\varsigma_k-\overline{\varsigma_k})\Big)\zeta^{-1}+\mathcal{O}(\zeta^{-2})$, and using the asymptotic estimates for $\{a_n,b_n,c_n,d_n\}_{n=1}^{m-1}$ given in Proposition 4.2, one forms the large-ζ asymptotics for $\widehat{\chi}(\zeta)$ given in Eq. (65) to show that

$$(\zeta(\widehat{\chi}(\zeta)(\delta(\zeta))^{\sigma_3}\prod_{k=m+1}^{N}(d_k^+(\zeta))^{\sigma_3}-\mathrm{I}))_{11}\underset{\zeta\in\mathbb{C}\setminus(\widehat{\sigma}_d\cup\sigma_c)}{\underset{\zeta\to\infty}{=}}(\widehat{\Delta}_o)_{11}+\widehat{a}_1^+-\widehat{a}_2^++a_m+\overline{d_m}+\sum_{k=m+1}^{N}(\varsigma_k-\overline{\varsigma_k})$$
$$+\,\mathrm{i}\left(\int_{-\infty}^{0}+\int_{\lambda_2}^{\lambda_1}\right)\ln(1-|r(\mu)|^2)\tfrac{\mathrm{d}\mu}{2\pi}+\mathcal{O}\left(\tfrac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\tfrac{\ln t}{t}\right)+\mathcal{O}\big(\mathrm{e}^{-\varrho t}\big)\,,$$
$$(\zeta(\widehat{\chi}(\zeta)(\delta(\zeta))^{\sigma_3}\prod_{k=m+1}^{N}(d_k^+(\zeta))^{\sigma_3}-\mathrm{I}))_{12}\underset{\zeta\in\mathbb{C}\setminus(\widehat{\sigma}_d\cup\sigma_c)}{\underset{\zeta\to\infty}{=}}(\widehat{\Delta}_o)_{12}+\widehat{a}_3^++b_m+\overline{c_m}+\frac{\sqrt{\nu(\lambda_1)}\,\mathrm{e}^{\frac{\mathrm{i}\Xi^+(0)}{2}}\lambda_1^{2\mathrm{i}\nu(\lambda_1)}}{\sqrt{t(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}$$
$$\times\left(\lambda_1\mathrm{e}^{-\mathrm{i}(\Theta^+(z_o,t)+\frac{\pi}{4})}+\lambda_2\mathrm{e}^{\mathrm{i}(\Theta^+(z_o,t)+\frac{\pi}{4})}\right)+\mathcal{O}\left(\tfrac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\tfrac{\ln t}{t}\right)+\mathcal{O}\big(\mathrm{e}^{-\varrho t}\big)\,,$$

where $\varrho:=4\min_{\substack{m\in\{1,2,\dots,N\}\\ n\neq m\in\{1,2,\dots,N\}}}\{\sin(\phi_n)|\cos(\phi_n)-\cos(\phi_m)|\}$ (>0). Neglecting exponentially small terms (cf. Remark 4.1), from the expressions for $u(x,t)$ and $\int_{+\infty}^{x}(|u(x',t)|^2-1)\,\mathrm{d}x'$ given, respectively, in Eqs. (63) and (64), and the trace identity (cf. Eq. (4)) $\int_{-\infty}^{+\infty}(|u(x',t)|^2-1)\,\mathrm{d}x'=(\int_{-\infty}^{x}+\int_{x}^{+\infty})(|u(x',t)|^2-1)\,\mathrm{d}x'=-2\sum_{n=1}^{N}\sin(\phi_n)-\int_{-\infty}^{+\infty}\ln(1-|r(\mu)|^2)\,\frac{\mathrm{d}\mu}{2\pi}$, one obtains the results stated in the Proposition. □

Proposition 4.6. *As* $t \to +\infty$ *and* $x \to -\infty$ *such that* $z_o := x/t < -2$ *and* $(x,t) \in \urcorner_m$, $m \in \{1,2,\dots,N\}$, *for* $\theta_{\gamma_m} = \pm\pi/2$,

$$
\begin{aligned}
(\widehat{\Delta}_o)_{11} =& \, \mathrm{i}\Bigg(\pm \frac{2\sin(\phi_m)|\gamma_m|P(\phi_m,\phi_k)Q(\phi_m)\mathrm{e}^{\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})} + \frac{\sqrt{\nu(\lambda_1)}}{\sqrt{t(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\big(-2\cos(\Theta^+(z_o,t) \\
&+ \tfrac{\pi}{4})\cos(s^+) + \frac{4\sin(\phi_m)|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\sin(s^+-\phi_m)\cos(\Theta^+(z_o,t)+\frac{\pi}{4})\mathrm{e}^{2\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})} \\
&+ \frac{8\lambda_1^2\sin^2(\phi_m)|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)(1+|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})\mathrm{e}^{2\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})^2(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)^2} \\
&\times \big(((\lambda_1+\lambda_2)\cos(\phi_m)-2)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4}) + (\lambda_1-\lambda_2)\sin(\phi_m)\sin(\Theta^+(z_o,t)+\tfrac{\pi}{4})\big)\cos(s^+) \\
&+ \frac{4\lambda_1\sin(\phi_m)\cos(\phi_m)|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)}\big(2\sin(s^+-\phi_m)\cos(\Theta^+(z_o,t) \\
&+ \tfrac{\pi}{4}) - (\lambda_1+\lambda_2)\sin(s^+)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4}) + (\lambda_1-\lambda_2)\cos(s^+)\sin(\Theta^+(z_o,t)+\tfrac{\pi}{4})\big)\big)\Bigg) \\
&+ \mathcal{O}\Big(\tfrac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\tfrac{\ln t}{t}\Big), \\
(\widehat{\Delta}_o)_{12} =& -\mathrm{i}\mathrm{e}^{-\mathrm{i}(\theta^+(1)+s^+)} + \frac{2\sin(\phi_m)|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{-\mathrm{i}(\theta^+(1)+\phi_m+s^+)+2\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})} \\
&+ \frac{1}{\sqrt{t}}\Bigg(\frac{2\mathrm{i}\mathrm{Im}(\widehat{a}_1-\widehat{a}_2)\sin(\phi_m)|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{-\mathrm{i}(\theta^+(1)+\phi_m+s^+)+2\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})} \\
&\pm \frac{4\mathrm{i}\sin(\phi_m)|\gamma_m|P(\phi_m,\phi_k)Q(\phi_m)\sqrt{\nu(\lambda_1)}\,\mathrm{e}^{-\mathrm{i}(\theta^+(1)+2s^+)+\phi(x,t)}\cos(\Theta^+(z_o,t)+\frac{\pi}{4})}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}} \\
&\mp \frac{16\mathrm{i}\lambda_1^2\sin^2(\phi_m)|\gamma_m|^3P^3(\phi_m,\phi_k)Q^3(\phi_m)\sqrt{\nu(\lambda_1)}\,\mathrm{e}^{-\mathrm{i}(\theta^+(1)+s^+)+3\phi(x,t)}\cos(s^+)}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})^2(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)^2\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}} \\
&\times \big(((\lambda_1+\lambda_2)\cos(\phi_m)-2)\sin(\phi_m)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4}) + (\lambda_1-\lambda_2)\sin^2(\phi_m)\sin(\Theta^+(z_o,t)+\tfrac{\pi}{4}) \\
&+ \mathrm{i}(((\lambda_1+\lambda_2)\cos(\phi_m)-2)\cos(\phi_m)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4}) + (\lambda_1-\lambda_2)\sin(\phi_m)\cos(\phi_m) \\
&\times \sin(\Theta^+(z_o,t)+\tfrac{\pi}{4})\big) + \mathrm{Im}(\widehat{a}_1-\widehat{a}_2)\mathrm{e}^{-\mathrm{i}(\theta^+(1)+s^+)} \\
&- \frac{4\lambda_1\sin(\phi_m)|\gamma_m|P(\phi_m,\phi_k)Q(\phi_m)\sqrt{\nu(\lambda_1)}\,\mathrm{e}^{-\mathrm{i}(\theta^+(1)+s^+)+\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}} \\
&\times \big(\mp 2\sin(s^+-\phi_m)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4}) \pm (\lambda_1+\lambda_2)\sin(s^+)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4}) \\
&\mp (\lambda_1-\lambda_2)\cos(s^+)\sin(\Theta^+(z_o,t)+\tfrac{\pi}{4})\big)\Bigg) + \mathcal{O}\Big(\tfrac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\tfrac{\ln t}{t}\Big),
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Im}(\widehat{a}_1-\widehat{a}_2) =& \pm \frac{\sqrt{\nu(\lambda_1)}\sin(s^+)\cos(\Theta^+(z_o,t)+\frac{\pi}{4})(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})}{\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}\sin(\phi_m)|\gamma_m|P(\phi_m,\phi_k)Q(\phi_m)\mathrm{e}^{\phi(x,t)}} \\
&\pm \frac{4\lambda_1^2\sqrt{\nu(\lambda_1)}\sin(\phi_m)|\gamma_m|P(\phi_m,\phi_k)Q(\phi_m)\sin(s^+)\mathrm{e}^{\phi(x,t)}}{(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)^2\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}} \\
&\times \big(((\lambda_1+\lambda_2)\cos(\phi_m)-2)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4}) + (\lambda_1-\lambda_2)\sin(\phi_m)\sin(\Theta^+(z_o,t)+\tfrac{\pi}{4})\big) \\
&\pm \frac{2\lambda_1\sqrt{\nu(\lambda_1)}\cos(\phi_m)|\gamma_m|P(\phi_m,\phi_k)Q(\phi_m)\mathrm{e}^{\phi(x,t)}}{(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\big(2\cos(s^+-\phi_m)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4}) \\
&- (\lambda_1+\lambda_2)\cos(s^+)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4}) - (\lambda_1-\lambda_2)\sin(s^+)\sin(\Theta^+(z_o,t)+\tfrac{\pi}{4})\big) \\
&\pm \frac{2\sqrt{\nu(\lambda_1)}|\gamma_m|P(\phi_m,\phi_k)Q(\phi_m)\cos(s^+-\phi_m)\cos(\Theta^+(z_o,t)+\frac{\pi}{4})\mathrm{e}^{\phi(x,t)}}{\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}} \\
&+ \mathcal{O}\Big(\tfrac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\tfrac{\ln t}{t}\Big),
\end{aligned}
$$

$$
\mathrm{Re}(\widehat{a}_1-\widehat{a}_2) = \mathrm{Re}(\widehat{a}_3) = \mathrm{Im}(\widehat{a}_3) = 0,
$$

where $\theta^+(\cdot)$ *is given in Theorem* 2.2.1, *Eq.* (8).

Proof. Recall from Lemma 4.1 that: (1) $\widehat{\Delta}_o=\widehat{\mathcal{P}}(0)\widehat{m}_d(0)\chi^c(0)(\delta(0))^{\sigma_3}\Big(\prod_{k=m+1}^N(d_k^+(0))^{\sigma_3}\Big)\sigma_2$; (2) $\mathrm{tr}(\widehat{\Delta}_o)=0$; (3) $\det(\widehat{\Delta}_o)=-1$; and (4) $\widehat{\Delta}_o\widehat{\Delta}_o=\mathrm{I}$. Taking the determinant of both sides of the above expression for $\widehat{\Delta}_o$ and using the fact that $\det(\widehat{\Delta}_o)=-1$, it follows that, modulo terms that are $\mathcal{O}(\frac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\frac{\ln t}{t})$, and always ignoring exponentially small terms, $\det(\widehat{\mathcal{P}}(0))=(\det(\widehat{m}_d(0)))^{-1}$. Before proceeding further, this will be verified; in particular, since $\widehat{m}_d(\zeta)\in\mathrm{SL}(2,\mathbb{C})$, it must be the case that, modulo terms that are $\mathcal{O}(\frac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\frac{\ln t}{t})$, $\det(\widehat{m}_d(0))=1$. From Lemma 4.3, keeping only leading-order terms, one shows that $\det(\widehat{\mathcal{P}}(0))=1+\frac{2\mathrm{Re}(\widehat{a}_1-\widehat{a}_2)}{\sqrt{t}}+\mathcal{O}(\frac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\frac{\ln t}{t})$, and, from the proof of Proposition 4.1, the estimates of Proposition 4.2, and noting that $(1-\frac{a_m^0}{\varsigma_m})(1-\overline{\frac{a_m^0}{\varsigma_m}})-\frac{c_m^0}{\varsigma_m}\overline{\frac{c_m^0}{\varsigma_m}}=1$ and $\int_{-\infty}^{+\infty}\frac{(1-\mu^2)\ln(1-|r(\mu)|^2)}{(\mu^2-2\mu\cos(\phi_m)+1)}\frac{\mathrm{d}\mu}{\mu}=0$ (which is proven using the symmetry reduction $r(\zeta^{-1})=-\overline{r(\overline{\zeta})}$), one shows that $(\det(\widehat{m}_d(0)))^{-1}=1+\frac{2}{\sqrt{t}}\mathrm{Re}\Big((1-\overline{\frac{a_m^0}{\varsigma_m}})(\frac{a_m^1}{\varsigma_m}+\overline{\frac{d_m^1}{\varsigma_m}})+\frac{c_m^0}{\varsigma_m}(\frac{b_m^1}{\varsigma_m}+\overline{\frac{c_m^1}{\varsigma_m}})\Big)+\mathcal{O}(\frac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\frac{\ln t}{t})$; thus, from the—yet to be verified—identity $\det(\widehat{\mathcal{P}}(0))=(\det(\widehat{m}_d(0)))^{-1}$, and the above, it follows that $\mathrm{Re}(\widehat{a}_1-\widehat{a}_2)=\mathrm{Re}\Big((1-\overline{\frac{a_m^0}{\varsigma_m}})(\frac{a_m^1}{\varsigma_m}+\overline{\frac{d_m^1}{\varsigma_m}})+\frac{c_m^0}{\varsigma_m}(\frac{b_m^1}{\varsigma_m}+\overline{\frac{c_m^1}{\varsigma_m}})\Big)$. If the formulae presented thus far are correct, then one must be able to show from them that the right-hand side of the latter relation equals zero. From Proposition 4.2, and repeated application of standard trigonometric identities, one shows, after a very lengthy and tedious algebraic calculation, that ($\theta_{\gamma_m}=\pm\pi/2$)

$$\begin{aligned}\mathrm{Re}\Big(\frac{c_m^0}{\varsigma_m}(\frac{b_m^1}{\varsigma_m}+\overline{\frac{c_m^1}{\varsigma_m}})\Big)&=\pm\frac{16\lambda_1^2\sin^3(\phi_m)|\gamma_m|^3P^3(\phi_m,\phi_k)Q^3(\phi_m)\sqrt{\nu(\lambda_1)}\,(1+|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})\mathrm{e}^{3\phi(x,t)}}{(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)^2\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})^3}\\&\times\Big(((\lambda_1+\lambda_2)\cos(\phi_m)-2)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})+(\lambda_1-\lambda_2)\sin(\phi_m)\sin(\Theta^+(z_o,t)+\tfrac{\pi}{4})\Big)\\&\times\cos(s^+)\pm\frac{8\lambda_1\sin^2(\phi_m)\cos(\phi_m)|\gamma_m|^3P^3(\phi_m,\phi_k)Q^3(\phi_m)\sqrt{\nu(\lambda_1)}\,\mathrm{e}^{3\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})^2(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\\&\times\Big(2\sin(s^+-\phi_m)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})-(\lambda_1+\lambda_2)\sin(s^+)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})\\&+(\lambda_1-\lambda_2)\cos(s^+)\sin(\Theta^+(z_o,t)+\tfrac{\pi}{4})\Big),\end{aligned}$$

and

$$\begin{aligned}\mathrm{Re}\Big((1-\overline{\frac{a_m^0}{\varsigma_m}})(\frac{a_m^1}{\varsigma_m}+\overline{\frac{d_m^1}{\varsigma_m}})\Big)&=\mp\frac{16\lambda_1^2\sin^3(\phi_m)|\gamma_m|^3P^3(\phi_m,\phi_k)Q^3(\phi_m)\sqrt{\nu(\lambda_1)}\,(1+|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})\mathrm{e}^{3\phi(x,t)}}{(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)^2\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})^3}\\&\times\Big(((\lambda_1+\lambda_2)\cos(\phi_m)-2)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})+(\lambda_1-\lambda_2)\sin(\phi_m)\sin(\Theta^+(z_o,t)+\tfrac{\pi}{4})\Big)\\&\times\cos(s^+)\mp\frac{8\lambda_1\sin^2(\phi_m)\cos(\phi_m)|\gamma_m|^3P^3(\phi_m,\phi_k)Q^3(\phi_m)\sqrt{\nu(\lambda_1)}\,\mathrm{e}^{3\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})^2(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\\&\times\Big(2\sin(s^+-\phi_m)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})-(\lambda_1+\lambda_2)\sin(s^+)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})\\&+(\lambda_1-\lambda_2)\cos(s^+)\sin(\Theta^+(z_o,t)+\tfrac{\pi}{4})\Big);\end{aligned}$$

thus, adding, $\mathrm{Re}\Big((1-\overline{\frac{a_m^0}{\varsigma_m}})(\frac{a_m^1}{\varsigma_m}+\overline{\frac{d_m^1}{\varsigma_m}})+\frac{c_m^0}{\varsigma_m}(\frac{b_m^1}{\varsigma_m}+\overline{\frac{c_m^1}{\varsigma_m}})\Big)=0$, whence $\mathrm{Re}(\widehat{a}_1-\widehat{a}_2)=0$. Recalling the expression for $(\widehat{\Delta}_o)_{11}$ given in Proposition 4.3, the estimates and expansions of Proposition 4.2, and the fact—just established—that $\mathrm{Re}(\widehat{a}_1-\widehat{a}_2)=0$, one shows that

$$(\widehat{\Delta}_o)_{11}=\frac{1}{\sqrt{t}}(\widehat{\Delta}_o)_{11}^{\alpha}+\mathrm{i}\Big((\widehat{\Delta}_o)_{11}^{\beta}+\frac{1}{\sqrt{t}}(\widehat{\Delta}_o)_{11}^{\gamma}\Big)+\mathcal{O}\Big(\frac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\frac{\ln t}{t}\Big),$$

where

$$\begin{aligned}(\widehat{\Delta}_o)_{11}^{\alpha}:=&\mp\frac{2\mathrm{Im}(\widehat{a}_1-\widehat{a}_2)\sin(\phi_m)|\gamma_m|P(\phi_m,\phi_k)Q(\phi_m)\mathrm{e}^{\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})}\\&+\frac{2\sqrt{\nu(\lambda_1)}\cos(\Theta^+(z_o,t)+\frac{\pi}{4})\sin(s^+)}{\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}-\mathrm{Re}(\widehat{a}_3)\sin(\theta^+(1)+s^+)-\mathrm{Im}(\widehat{a}_3)\cos(\theta^+(1)+s^+)\\&+\frac{4\sin(\phi_m)|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\sqrt{\nu(\lambda_1)}\cos(\Theta^+(z_o,t)+\frac{\pi}{4})\cos(s^+-\phi_m)\mathrm{e}^{2\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\\&+\frac{2\mathrm{Re}(\widehat{a}_3)\sin(\phi_m)|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\cos(s^++\phi_m+\theta^+(1))\mathrm{e}^{2\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})}\\&-\frac{2\mathrm{Im}(\widehat{a}_3)\sin(\phi_m)|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\sin(s^++\phi_m+\theta^+(1))\mathrm{e}^{2\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})}\\&+\frac{8\lambda_1^2\sin^2(\phi_m)|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\sqrt{\nu(\lambda_1)}\sin(s^+)\mathrm{e}^{2\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)^2\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\end{aligned}$$

$$\times\left(((\lambda_1+\lambda_2)\cos(\phi_m)-2)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})+(\lambda_1-\lambda_2)\sin(\phi_m)\sin(\Theta^+(z_o,t)+\tfrac{\pi}{4})\right)$$
$$+\frac{4\lambda_1\sin(\phi_m)\cos(\phi_m)|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\sqrt{\nu(\lambda_1)}\,\mathrm{e}^{2\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}$$
$$\times\left(2\cos(s^+-\phi_m)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})-(\lambda_1+\lambda_2)\cos(s^+)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})\right.$$
$$\left.-(\lambda_1-\lambda_2)\sin(s^+)\sin(\Theta^+(z_o,t)+\tfrac{\pi}{4})\right),$$

$$(\widehat{\Delta}_o)_{11}^{\beta}:=\pm\frac{2\sin(\phi_m)|\gamma_m|P(\phi_m,\phi_k)Q(\phi_m)\mathrm{e}^{\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})},$$

$$(\widehat{\Delta}_o)_{11}^{\gamma}:=-\frac{2\sqrt{\nu(\lambda_1)}\cos(\Theta^+(z_o,t)+\frac{\pi}{4})\cos(s^+)}{\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}+\mathrm{Re}(\widehat{a}_3)\cos(\theta^+(1)+s^+)-\mathrm{Im}(\widehat{a}_3)\sin(\theta^+(1)+s^+)$$
$$+\frac{4\sin(\phi_m)|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\sqrt{\nu(\lambda_1)}\cos(\Theta^+(z_o,t)+\frac{\pi}{4})\sin(s^+-\phi_m)\mathrm{e}^{2\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}$$
$$+\frac{2\mathrm{Re}(\widehat{a}_3)\sin(\phi_m)|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\sin(s^++\phi_m+\theta^+(1))\mathrm{e}^{2\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})}$$
$$+\frac{2\mathrm{Im}(\widehat{a}_3)\sin(\phi_m)|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\cos(s^++\phi_m+\theta^+(1))\mathrm{e}^{2\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})}$$
$$+\frac{8\lambda_1^2\sin^2(\phi_m)|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\sqrt{\nu(\lambda_1)}\cos(s^+)(1+|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})\mathrm{e}^{2\phi(x,t)}}{(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)^2\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}(1-|\gamma|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})^2}$$
$$\times\left(((\lambda_1+\lambda_2)\cos(\phi_m)-2)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})+(\lambda_1-\lambda_2)\sin(\phi_m)\sin(\Theta^+(z_o,t)+\tfrac{\pi}{4})\right)$$
$$+\frac{4\lambda_1\sin(\phi_m)\cos(\phi_m)|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\sqrt{\nu(\lambda_1)}\,\mathrm{e}^{2\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}$$
$$\times\left(2\sin(s^+-\phi_m)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})-(\lambda_1+\lambda_2)\sin(s^+)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})\right.$$
$$\left.+(\lambda_1-\lambda_2)\cos(s^+)\sin(\Theta^+(z_o,t)+\tfrac{\pi}{4})\right),$$

and $\theta^+(\cdot)$ is specified in the Proposition. Recalling that $\mathrm{tr}(\widehat{\Delta}_o)=0$, it follows that $\mathrm{Re}((\widehat{\Delta}_o)_{11})=0$; thus, $(\widehat{\Delta}_o)_{11}^{\alpha}=0$, which gives a relation for $\mathrm{Im}(\widehat{a}_1-\widehat{a}_2)$, but, since $\mathrm{Re}(\widehat{a}_3)$ and $\mathrm{Im}(\widehat{a}_3)$ are as yet undetermined, this is not enough. Towards this end, one uses the condition $\det(\widehat{\Delta}_o)=(\widehat{\Delta}_o)_{11}\overline{(\widehat{\Delta}_o)_{11}}-(\widehat{\Delta}_o)_{12}\overline{(\widehat{\Delta}_o)_{12}}=-1$ (Note: if the conditions $\mathrm{tr}(\widehat{\Delta}_o)=0$ and $\det(\widehat{\Delta}_o)=-1$ are satisfied, then it follows that $\widehat{\Delta}_o\overline{\widehat{\Delta}_o}=\mathrm{I}$ is also satisfied, so it is enough to use the condition $\det(\widehat{\Delta}_o)=-1$). From the formula for $(\widehat{\Delta}_o)_{11}$ given above, and the expression for $(\widehat{\Delta}_o)_{12}$ given in Proposition 4.3, one shows that

$$(\widehat{\Delta}_o)_{12}\overline{(\widehat{\Delta}_o)_{12}}=(1-\tfrac{a_m^0}{\varsigma_m})(1-\tfrac{\overline{a_m^0}}{\overline{\varsigma_m}})+\tfrac{2}{\sqrt{t}}\Big(\mathrm{Re}\Big((\widehat{a}_1-\widehat{a}_2)(1-\tfrac{a_m^0}{\varsigma_m})(1-\tfrac{\overline{a_m^0}}{\overline{\varsigma_m}})\Big)$$
$$-\mathrm{Re}\Big((1-\tfrac{\overline{a_m^0}}{\overline{\varsigma_m}})(\tfrac{a_m^1}{\varsigma_m}+\tfrac{\overline{d_m^1}}{\overline{\varsigma_m}})\Big)+\mathrm{Re}\left((1-\tfrac{a_m^0}{\varsigma_m})\frac{2c_m^0\sqrt{\nu(\lambda_1)}\,\delta(0)\cos(\Theta^+(z_o,t)+\frac{\pi}{4})}{\varsigma_m\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\right)$$
$$-\mathrm{Re}\Big((1-\tfrac{\overline{a_m^0}}{\overline{\varsigma_m}})\tfrac{c_m^0\widehat{a}_3}{\varsigma_m}\Big)\Big)+\mathcal{O}\Big(\tfrac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\tfrac{\ln t}{t}\Big).$$

Using the estimates given in Proposition 4.2, and recalling that $\mathrm{Re}(\widehat{a}_1-\widehat{a}_2)=0$, one gets that

$$(\widehat{\Delta}_o)_{12}\overline{(\widehat{\Delta}_o)_{12}}=1+\frac{4\sin^2(\phi_m)|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})}+\frac{4\sin^2(\phi_m)|\gamma_m|^4P^4(\phi_m,\phi_k)Q^4(\phi_m)\mathrm{e}^{4\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})^2}$$
$$+\frac{2}{\sqrt{t}}\Bigg(\mp\frac{4\sin(\phi_m)|\gamma_m|P(\phi_m,\phi_k)Q(\phi_m)\sqrt{\nu(\lambda_1)}\,\cos(s^+)\cos(\Theta^+(z_o,t)+\frac{\pi}{4})\mathrm{e}^{\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}$$
$$\pm\frac{8\sin^2(\phi_m)|\gamma_m|^3P^3(\phi_m,\phi_k)Q^3(\phi_m)\sqrt{\nu(\lambda_1)}\,\sin(s^+-\phi_m)\cos(\Theta^+(z_o,t)+\frac{\pi}{4})\mathrm{e}^{3\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})^2\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}$$
$$\pm\frac{2\sin(\phi_m)|\gamma_m|P(\phi_m,\phi_k)Q(\phi_m)\mathrm{e}^{\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})}\left(\mathrm{Re}(\widehat{a}_3)\cos(s^++\theta^+(1))-\mathrm{Im}(\widehat{a}_3)\sin(s^++\theta^+(1))\right)$$
$$\pm\frac{4\sin^2(\phi_m)|\gamma_m|^3P^3(\phi_m,\phi_k)Q^3(\phi_m)\mathrm{e}^{3\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})^2}\left(\mathrm{Re}(\widehat{a}_3)\sin(s^++\theta^+(1)+\phi_m)+\mathrm{Im}(\widehat{a}_3)\cos(s^++\theta^+(1)+\phi_m)\right)$$
$$\pm\frac{16\lambda_1^2\sin^3(\phi_m)|\gamma_m|^3P^3(\phi_m,\phi_k)Q^3(\phi_m)\sqrt{\nu(\lambda_1)}\,\cos(s^+)(1+|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})\mathrm{e}^{3\phi(x,t)}}{(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)^2\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})^3}$$
$$\times\left(((\lambda_1+\lambda_2)\cos(\phi_m)-2)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})+(\lambda_1-\lambda_2)\sin(\phi_m)\sin(\Theta^+(z_o,t)+\tfrac{\pi}{4})\right)$$

$$
\begin{aligned}
&\pm\,\frac{8\lambda_1\sin^2(\phi_m)\cos(\phi_m)|\gamma_m|^3P^3(\phi_m,\phi_k)Q^3(\phi_m)\sqrt{\nu(\lambda_1)}\,\mathrm{e}^{3\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})^2(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\\
&\times\,\big(2\sin(s^+-\phi_m)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})-(\lambda_1+\lambda_2)\sin(s^+)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})\\
&+\,(\lambda_1-\lambda_2)\cos(s^+)\sin(\Theta^+(z_o,t)+\tfrac{\pi}{4})\big)\big)+\mathcal{O}\Big(\tfrac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\tfrac{\ln t}{t}\Big).
\end{aligned}
$$

From the expression for $(\widehat{\Delta}_o)_{11}$ given above, and using the fact that $(\widehat{\Delta}_o)_{11}^{\alpha}=0$, one shows that

$$
\begin{aligned}
(\widehat{\Delta}_o)_{11}\overline{(\widehat{\Delta}_o)_{11}}&=\frac{4\sin^2(\phi_m)|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})^2}\\
&+\,\frac{2}{\sqrt{t}}\bigg(\mp\frac{4\sin(\phi_m)|\gamma_m|P(\phi_m,\phi_k)Q(\phi_m)\sqrt{\nu(\lambda_1)}\,\cos(s^+)\cos(\Theta^+(z_o,t)+\frac{\pi}{4})\mathrm{e}^{\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\\
&\pm\,\frac{8\sin^2(\phi_m)|\gamma_m|^3P^3(\phi_m,\phi_k)Q^3(\phi_m)\sqrt{\nu(\lambda_1)}\,\sin(s^+-\phi_m)\cos(\Theta^+(z_o,t)+\frac{\pi}{4})\mathrm{e}^{3\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})^2\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\\
&\pm\,\frac{2\mathrm{Re}(\widehat{a}_3)\sin(\phi_m)|\gamma_m|P(\phi_m,\phi_k)Q(\phi_m)\mathrm{e}^{\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})}\big(\cos(s^++\theta^+(1))\\
&+\,\frac{2\sin(\phi_m)|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\sin(s^++\phi_m+\theta^+(1))\mathrm{e}^{2\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})}\bigg)\\
&\pm\,\frac{2\mathrm{Im}(\widehat{a}_3)\sin(\phi_m)|\gamma_m|P(\phi_m,\phi_k)Q(\phi_m)\mathrm{e}^{\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})}\big(-\sin(s^++\theta^+(1))\\
&+\,\frac{2\sin(\phi_m)|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\cos(s^++\phi_m+\theta^+(1))\mathrm{e}^{2\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})}\bigg)\\
&\pm\,\frac{16\lambda_1^2\sin^3(\phi_m)|\gamma_m|^3P^3(\phi_m,\phi_k)Q^3(\phi_m)\sqrt{\nu(\lambda_1)}\,\cos(s^+)(1+|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})\mathrm{e}^{3\phi(x,t)}}{(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)^2\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})^3}\\
&\times\,\big(((\lambda_1+\lambda_2)\cos(\phi_m)-2)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})+(\lambda_1-\lambda_2)\sin(\phi_m)\sin(\Theta^+(z_o,t)+\tfrac{\pi}{4})\big)\\
&\pm\,\frac{8\lambda_1\sin^2(\phi_m)\cos(\phi_m)|\gamma_m|^3P^3(\phi_m,\phi_k)Q^3(\phi_m)\sqrt{\nu(\lambda_1)}\,\mathrm{e}^{3\phi(x,t)}}{(1-|\gamma_m|^2P^2(\phi_m,\phi_k)Q^2(\phi_m)\mathrm{e}^{2\phi(x,t)})^2(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\\
&\times\,\big(2\sin(s^+-\phi_m)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})-(\lambda_1+\lambda_2)\sin(s^+)\cos(\Theta^+(z_o,t)+\tfrac{\pi}{4})\\
&+\,(\lambda_1-\lambda_2)\cos(s^+)\sin(\Theta^+(z_o,t)+\tfrac{\pi}{4})\big)\big)+\mathcal{O}\Big(\tfrac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\tfrac{\ln t}{t}\Big).
\end{aligned}
$$

Now, taking note of the relation $(\widehat{\Delta}_o)_{11}^{\beta}\overline{(\widehat{\Delta}_o)_{11}^{\beta}}-(1-\frac{a_m^0}{\varsigma_m})(1-\overline{\frac{a_m^0}{\varsigma_m}})=-1$, one substitutes the above-derived formulae for $|(\widehat{\Delta}_o)_{11}|^2$ and $|(\widehat{\Delta}_o)_{12}|^2$ into $|(\widehat{\Delta}_o)_{11}|^2-|(\widehat{\Delta}_o)_{12}|^2=-1$, and, modulo terms that are $\mathcal{O}(\frac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\frac{\ln t}{t})$, gets **exact** cancellation at $\mathcal{O}(1)$ and $\mathcal{O}(t^{-1/2})$; thus, one concludes that $\mathrm{Re}(\widehat{a}_3)=\mathrm{Im}(\widehat{a}_3)=0$. Recalling that $(\widehat{\Delta}_o)_{11}^{\alpha}=0$, and using the fact that $\mathrm{Re}(\widehat{a}_3)=\mathrm{Im}(\widehat{a}_3)=0$, from the expression for $(\widehat{\Delta}_0)_{11}$ given above, one obtains, after some straightforward algebra, the expressions for $\mathrm{Im}(\widehat{a}_1-\widehat{a}_2)$ and $(\widehat{\Delta}_o)_{11}$ stated in the Proposition. From Proposition 4.2, and the fact that $\mathrm{Re}(\widehat{a}_3)=\mathrm{Im}(\widehat{a}_3)=\mathrm{Re}(\widehat{a}_1-\widehat{a}_2)=0$, one obtains, upon recalling the expression for $(\widehat{\Delta}_o)_{12}$ given in Proposition 4.3, the formula for $(\widehat{\Delta}_o)_{12}$ given in the Proposition. □

Lemma 4.4. *As $t\to+\infty$ and $x\to-\infty$ such that $z_o:=x/t<-2$ and $(x,t)\in\urcorner_m$, $m\in\{1,2,\dots,N\}$, $u(x,t)$, the solution of the Cauchy problem for the* D_fNLSE, *and $\int_{\pm\infty}^{x}(|u(x',t)|^2-1)\,\mathrm{d}x'$ have the leading-order asymptotic expansions (for the upper sign) stated in Theorem* 2.2.1, *Eqs.* (7)–(20).

Proof. The asymptotic expansions for $u(x,t)$ and $\int_{\pm\infty}^{x}(|u(x',t)|^2-1)\,\mathrm{d}x'$ follow from Proposition 4.2, Proposition 4.4, Eqs. (70)–(72), and Proposition 4.6 after tedious, but otherwise straightforward algebraic calculations. □

Appendix A. Asymptotic Analysis as $t \to -\infty$

In this appendix, a silhouette of the asymptotic analysis for $u(x,t)$ and $\int_{\pm\infty}^{x}(|u(x',t)|^2-1)\,\mathrm{d}x'$ as $t\to-\infty$ and $x\to+\infty$ such that $z_o:=x/t<-2$ and $(x,t)\in \daleth_m$, $m\in\{1,2,\ldots,N\}$, is presented. Since the calculations are analogous to those of Sections 3 and 4, only final results/statements, with in one instance a sketch of a proof, are given: one mimics the scheme of the calculation in Sections 3 and 4 to arrive at the corresponding asymptotic results.

The analogue of Lemma 3.1 is

Lemma A.1.1. *For* $r(\zeta)\in\mathcal{S}_{\mathbb{C}}^{1}(\mathbb{R})$*, let* $m(\zeta)\colon \mathbb{C}\setminus(\sigma_d\cup\sigma_c)\to \mathrm{M}_2(\mathbb{C})$ *be the solution of the* RHP *formulated in Lemma* 2.1.2. *Set* $\widetilde{m}(\zeta):=m(\zeta)(\widetilde{\delta}(\zeta))^{-\sigma_3}$*, where* $\widetilde{\delta}(\zeta)=\exp\Big(\Big(\int_0^{\lambda_2}+\int_{\lambda_1}^{+\infty}\Big)\frac{\ln(1-|r(\mu)|^2)}{(\mu-\zeta)}\frac{\mathrm{d}\mu}{2\pi\mathrm{i}}\Big)$*, with* λ_1 *and* λ_2 *given in Theorem* 2.2.1*, Eq.* (10)*,* $\widetilde{\delta}(\zeta)\overline{\widetilde{\delta}(\overline{\zeta})}=1$*,* $\widetilde{\delta}(\zeta)\widetilde{\delta}(\zeta^{-1})=\widetilde{\delta}(0)$*, and* $||(\widetilde{\delta}(\cdot))^{\pm1}||_{\mathcal{L}^\infty(\mathbb{C})}:=\sup_{\zeta\in\mathbb{C}}|(\widetilde{\delta}(\zeta))^{\pm1}|<\infty$*. Then* $\widetilde{m}(\zeta)\colon \mathbb{C}\setminus(\sigma_d\cup\sigma_c)\to\mathrm{M}_2(\mathbb{C})$ *solves the following* RHP:

(i) $\widetilde{m}(\zeta)$ *is piecewise (sectionally) meromorphic* $\forall\,\zeta\in\mathbb{C}\setminus\sigma_c$;

(ii) $\widetilde{m}_{\pm}(\zeta):=\lim_{\substack{\zeta'\to\zeta\\ \pm\mathrm{Im}(\zeta')>0}}\widetilde{m}(\zeta')$ *satisfy the jump condition*

$$\widetilde{m}_{+}(\zeta)=\widetilde{m}_{-}(\zeta)\exp(-\mathrm{i}k(\zeta)(x+2\lambda(\zeta)t)\mathrm{ad}(\sigma_3))\widetilde{\mathcal{G}}(\zeta),\quad \zeta\in\mathbb{R},$$

where

$$\widetilde{\mathcal{G}}(\zeta)=\begin{pmatrix}(1-r(\zeta)\overline{r(\overline{\zeta})})\widetilde{\delta}_{-}(\zeta)(\widetilde{\delta}_{+}(\zeta))^{-1} & -\overline{r(\overline{\zeta})}\,\widetilde{\delta}_{-}(\zeta)\widetilde{\delta}_{+}(\zeta)\\ r(\zeta)(\widetilde{\delta}_{-}(\zeta)\widetilde{\delta}_{+}(\zeta))^{-1} & (\widetilde{\delta}_{-}(\zeta))^{-1}\widetilde{\delta}_{+}(\zeta)\end{pmatrix};$$

(iii) $\widetilde{m}(\zeta)$ *has simple poles in* $\sigma_d=\cup_{n=1}^{N}(\{\varsigma_n\}\cup\{\overline{\varsigma_n}\})$ *with*

$$\begin{aligned}\mathrm{Res}(\widetilde{m}(\zeta);\varsigma_n)&=\lim_{\zeta\to\varsigma_n}\widetilde{m}(\zeta)g_n(\widetilde{\delta}(\varsigma_n))^{-2}\sigma_{-}, && n\in\{1,2,\ldots,N\},\\ \mathrm{Res}(\widetilde{m}(\zeta);\overline{\varsigma_n})&=\sigma_1\overline{\mathrm{Res}(\widetilde{m}(\zeta);\varsigma_n)}\,\sigma_1, && n\in\{1,2,\ldots,N\},\end{aligned}$$

where g_n *is defined in Lemma* 3.1*, (iii);*

(iv) $\det(\widetilde{m}(\zeta))|_{\zeta=\pm1}=0$;

(v) $\widetilde{m}(\zeta)\underset{\zeta\to0}{=}\zeta^{-1}(\widetilde{\delta}(0))^{\sigma_3}\sigma_2+\mathcal{O}(1)$;

(vi) $\widetilde{m}(\zeta)\underset{\substack{\zeta\to\infty\\ \zeta\in\mathbb{C}\setminus(\sigma_d\cup\sigma_c)}}{=}\mathrm{I}+\mathcal{O}(\zeta^{-1})$;

(vii) $\widetilde{m}(\zeta)=\sigma_1\overline{\widetilde{m}(\overline{\zeta})}\,\sigma_1$ *and* $\widetilde{m}(\zeta^{-1})=\zeta\widetilde{m}(\zeta)(\widetilde{\delta}(0))^{\sigma_3}\sigma_2$.

Let

$$u(x,t):=\mathrm{i}\lim_{\substack{\zeta\to\infty\\ \zeta\in\mathbb{C}\setminus(\sigma_d\cup\sigma_c)}}(\zeta(\widetilde{m}(\zeta)(\widetilde{\delta}(\zeta))^{\sigma_3}-\mathrm{I}))_{12},\tag{73}$$

and

$$\int_{+\infty}^{x}(|u(x',t)|^2-1)\,\mathrm{d}x':=-\mathrm{i}\lim_{\substack{\zeta\to\infty\\ \zeta\in\mathbb{C}\setminus(\sigma_d\cup\sigma_c)}}(\zeta(\widetilde{m}(\zeta)(\widetilde{\delta}(\zeta))^{\sigma_3}-\mathrm{I}))_{11}.\tag{74}$$

Then $u(x,t)$ *is the solution of the Cauchy problem for the* D_fNLSE.

The analogue of Definition 3.1 is

Definition A.1.1. *For* $m\in\{1,2,\ldots,N\}$ *and* $\{\varsigma_n\}_{n=1}^{m-1}\subset\mathbb{C}_+$ *(respectively,* $\{\overline{\varsigma_n}\}_{n=1}^{m-1}\subset\mathbb{C}_-$*), define the clockwise (respectively, counter-clockwise) oriented circles* $\widetilde{\mathfrak{K}}_n:=\{\zeta;\,|\zeta-\varsigma_n|=\widetilde{\varepsilon}_n^{\mathfrak{K}}\}$ *(respectively,* $\widetilde{\mathfrak{L}}_n:=\{\zeta;\,|\zeta-\overline{\varsigma_n}|=\widetilde{\varepsilon}_n^{\mathfrak{L}}\}$*), with* $\widetilde{\varepsilon}_n^{\mathfrak{K}}$ *(respectively,* $\widetilde{\varepsilon}_n^{\mathfrak{L}}$*) chosen sufficiently small such that* $\widetilde{\mathfrak{K}}_n\cap\widetilde{\mathfrak{K}}_{n'}=\widetilde{\mathfrak{L}}_n\cap\widetilde{\mathfrak{L}}_{n'}=\widetilde{\mathfrak{K}}_n\cap\widetilde{\mathfrak{L}}_n=\widetilde{\mathfrak{K}}_n\cap\sigma_c=\widetilde{\mathfrak{L}}_n\cap\sigma_c=\emptyset\ \forall\ n\neq n'\in\{1,2,\ldots,m-1\}$.

The analogue of Lemma 3.2 is

Lemma A.1.2. *For $r(\zeta) \in \mathcal{S}_{\mathbb{C}}^{1}(\mathbb{R})$, let $\widetilde{m}(\zeta) \colon \mathbb{C} \setminus (\sigma_d \cup \sigma_c) \to \mathrm{M}_2(\mathbb{C})$ be the solution of the* RHP *formulated in Lemma* A.1.1. *Set*

$$\widetilde{m}^{\flat}(\zeta) := \begin{cases} \widetilde{m}(\zeta), & \zeta \in \mathbb{C} \setminus (\sigma_c \cup (\cup_{n=1}^{m-1}(\widetilde{\mathcal{K}}_n \cup \mathrm{int}(\widetilde{\mathcal{K}}_n) \cup \widetilde{\mathcal{L}}_n \cup \mathrm{int}(\widetilde{\mathcal{L}}_n)))), \\ \widetilde{m}(\zeta)\Big(\mathrm{I} - \frac{g_n(\widetilde{\delta}(\varsigma_n))^{-2}}{(\zeta - \varsigma_n)} \sigma_- \Big), & \zeta \in \mathrm{int}(\widetilde{\mathcal{K}}_n), \quad n \in \{1, 2, \dots, m-1\}, \\ \widetilde{m}(\zeta)\Bigg(\mathrm{I} + \frac{\overline{g_n(\widetilde{\delta}(\varsigma_n))^{-2}}}{(\zeta - \overline{\varsigma_n})} \sigma_+ \Bigg), & \zeta \in \mathrm{int}(\widetilde{\mathcal{L}}_n), \quad n \in \{1, 2, \dots, m-1\}. \end{cases}$$

Then $\widetilde{m}^{\flat}(\zeta) \colon \mathbb{C} \setminus ((\sigma_d \setminus \cup_{n=1}^{m-1}(\{\varsigma_n\} \cup \{\overline{\varsigma_n}\})) \cup (\sigma_c \cup (\cup_{n=1}^{m-1}(\widetilde{\mathcal{K}}_n \cup \widetilde{\mathcal{L}}_n)))) \to \mathrm{M}_2(\mathbb{C})$ solves the following RHP:

(i) $\widetilde{m}^{\flat}(\zeta)$ *is piecewise (sectionally) meromorphic* $\forall \zeta \in \mathbb{C} \setminus (\sigma_c \cup (\cup_{n=1}^{m-1}(\widetilde{\mathcal{K}}_n \cup \widetilde{\mathcal{L}}_n)))$;

(ii) $\widetilde{m}^{\flat}_{\pm}(\zeta) := \lim_{\substack{\zeta' \to \zeta \\ \zeta' \in \pm \,\text{side of}\, \sigma_c \cup (\cup_{n=1}^{m-1}(\widetilde{\mathcal{K}}_n \cup \widetilde{\mathcal{L}}_n))}} \widetilde{m}^{\flat}(\zeta')$ *satisfy the jump condition*

$$\widetilde{m}^{\flat}_{+}(\zeta) = \widetilde{m}^{\flat}_{-}(\zeta) \widetilde{v}^{\flat}(\zeta), \quad \zeta \in \sigma_c \cup (\cup_{n=1}^{m-1}(\widetilde{\mathcal{K}}_n \cup \widetilde{\mathcal{L}}_n)),$$

where

$$\widetilde{v}^{\flat}(\zeta) = \begin{cases} \exp(-\mathrm{i}k(\zeta)(x + 2\lambda(\zeta)t)\mathrm{ad}(\sigma_3))\widetilde{\mathcal{G}}(\zeta), & \zeta \in \mathbb{R}, \\ \mathrm{I} + \frac{g_n(\widetilde{\delta}(\varsigma_n))^{-2}}{(\zeta - \varsigma_n)} \sigma_-, & \zeta \in \widetilde{\mathcal{K}}_n, \quad n \in \{1, 2, \dots, m-1\}, \\ \mathrm{I} + \frac{\overline{g_n(\widetilde{\delta}(\varsigma_n))^{-2}}}{(\zeta - \overline{\varsigma_n})} \sigma_+, & \zeta \in \widetilde{\mathcal{L}}_n, \quad n \in \{1, 2, \dots, m-1\}, \end{cases}$$

with $\widetilde{\mathcal{G}}(\zeta)$ given in Lemma A.1.1, *(ii);*

(iii) $\widetilde{m}^{\flat}(\zeta)$ *has simple poles in* $\sigma_d \setminus \cup_{n=1}^{m-1}(\{\varsigma_n\} \cup \{\overline{\varsigma_n}\})$ *with*

$$\begin{aligned} \mathrm{Res}(\widetilde{m}^{\flat}(\zeta); \varsigma_n) &= \lim_{\zeta \to \varsigma_n} \widetilde{m}^{\flat}(\zeta) g_n(\widetilde{\delta}(\varsigma_n))^{-2} \sigma_-, && n \in \{m, m+1, \dots, N\}, \\ \mathrm{Res}(\widetilde{m}^{\flat}(\zeta); \overline{\varsigma_n}) &= \sigma_1 \overline{\mathrm{Res}(\widetilde{m}^{\flat}(\zeta); \varsigma_n)}\, \sigma_1, && n \in \{m, m+1, \dots, N\}; \end{aligned}$$

(iv) $\det(\widetilde{m}^{\flat}(\zeta))|_{\zeta = \pm 1} = 0$;

(v) $\widetilde{m}^{\flat}(\zeta) =_{\zeta \to 0} \zeta^{-1}(\widetilde{\delta}(0))^{\sigma_3} \sigma_2 + \mathcal{O}(1)$;

(vi) as $\zeta \to \infty$, $\zeta \in \mathbb{C} \setminus ((\sigma_d \setminus \cup_{n=1}^{m-1}(\{\varsigma_n\} \cup \{\overline{\varsigma_n}\})) \cup (\sigma_c \cup (\cup_{n=1}^{m-1}(\widetilde{\mathcal{K}}_n \cup \widetilde{\mathcal{L}}_n))))$, $\widetilde{m}^{\flat}(\zeta) = \mathrm{I} + \mathcal{O}(\zeta^{-1})$;

(vii) $\widetilde{m}^{\flat}(\zeta) = \sigma_1 \overline{\widetilde{m}^{\flat}(\overline{\zeta})}\, \sigma_1$ *and* $\widetilde{m}^{\flat}(\zeta^{-1}) = \zeta \widetilde{m}^{\flat}(\zeta)(\widetilde{\delta}(0))^{\sigma_3} \sigma_2$.

For $\zeta \in \mathbb{C} \setminus ((\sigma_d \setminus \cup_{n=1}^{m-1}(\{\varsigma_n\} \cup \{\overline{\varsigma_n}\})) \cup (\sigma_c \cup (\cup_{n=1}^{m-1}(\widetilde{\mathcal{K}}_n \cup \widetilde{\mathcal{L}}_n))))$, let

$$u(x,t) := \mathrm{i} \lim_{\zeta \to \infty} (\zeta(\widetilde{m}^{\flat}(\zeta)(\widetilde{\delta}(\zeta))^{\sigma_3} - \mathrm{I}))_{12}, \tag{75}$$

and

$$\int_{+\infty}^{x} (|u(x',t)|^2 - 1)\, \mathrm{d}x' := -\mathrm{i} \lim_{\zeta \to \infty} (\zeta(\widetilde{m}(\zeta)(\widetilde{\delta}(\zeta))^{\sigma_3} - \mathrm{I}))_{11}. \tag{76}$$

Then $u(x,t)$ is the solution of the Cauchy problem for the D_fNLSE.

The analogue of Lemma 3.3 is

Lemma A.1.3. *For $m \in \{1, 2, \dots, N\}$, let $\sigma''_d := \sigma_d \setminus \cup_{n=1}^{m-1}(\{\varsigma_n\} \cup \{\overline{\varsigma_n}\})$, $\sigma''_c := \sigma_c \cup (\cup_{n=1}^{m-1}(\widetilde{\mathcal{K}}_n \cup \widetilde{\mathcal{L}}_n))$, where $\widetilde{\mathcal{K}}_n$ and $\widetilde{\mathcal{L}}_n$ are given in Definition* A.1.1, *and $\sigma''_{\mathcal{O}\mathcal{D}} := \sigma''_d \cup \sigma''_c$ ($\sigma''_d \cap \sigma''_c = \emptyset$). Set*

$$\widetilde{m}^{\sharp}(\zeta) := \begin{cases} \widetilde{m}^{\flat}(\zeta) \prod_{k=1}^{m-1}(d_k^{+}(\zeta))^{-\sigma_3}, & \zeta \in \mathbb{C} \setminus (\sigma''_c \cup (\cup_{n=1}^{m-1}(\mathrm{int}(\widetilde{\mathcal{K}}_n) \cup \mathrm{int}(\widetilde{\mathcal{L}}_n)))), \\ \widetilde{m}^{\flat}(\zeta)(\widetilde{J}_{\widetilde{\mathcal{K}}_n}(\zeta))^{-1} \prod_{k=1}^{m-1}(d_k^{-}(\zeta))^{-\sigma_3}, & \zeta \in \mathrm{int}(\widetilde{\mathcal{K}}_n), \quad n \in \{1, 2, \dots, m-1\}, \\ \widetilde{m}^{\flat}(\zeta)(\widetilde{J}_{\widetilde{\mathcal{L}}_n}(\zeta))^{-1} \prod_{k=1}^{m-1}(d_k^{-}(\zeta))^{-\sigma_3}, & \zeta \in \mathrm{int}(\widetilde{\mathcal{L}}_n), \quad n \in \{1, 2, \dots, m-1\}, \end{cases}$$

where $d_n^{\pm}(\zeta)$ *are given in Lemma* 3.3, $\widetilde{J}_{\widetilde{\mathcal{K}}_n}(\zeta)$ $(\in \mathrm{SL}(2,\mathbb{C}))$ *and* $\widetilde{J}_{\widetilde{\mathcal{L}}_n}(\zeta)$ $(\in \mathrm{SL}(2,\mathbb{C}))$, *respectively, are holomorphic in* $\cup_{k=1}^{m-1}\mathrm{int}(\widetilde{\mathcal{K}}_k)$ *and* $\cup_{l=1}^{m-1}\mathrm{int}(\widetilde{\mathcal{L}}_l)$, *with*

$$\widetilde{J}_{\widetilde{\mathcal{K}}_n}(\zeta)=\begin{pmatrix} \dfrac{\prod_{\substack{k=1\\k\neq n}}^{m-1}\frac{d_k^+(\zeta)}{d_k^-(\zeta)}-\frac{\widetilde{C}_n^{\mathcal{K}}g_n(\widetilde{\delta}(\varsigma_n))^{-2}}{(\zeta-\overline{\varsigma_n})^2}\prod_{\substack{k=1\\k\neq n}}^{m-1}\frac{(d_k^+(\zeta))^{-1}}{d_k^-(\zeta)}}{(\zeta-\varsigma_n)} & \dfrac{\widetilde{C}_n^{\mathcal{K}}}{(\zeta-\overline{\varsigma_n})^2}\prod_{\substack{k=1\\k\neq n}}^{m-1}\dfrac{(d_k^+(\zeta))^{-1}}{d_k^-(\zeta)} \\ -g_n(\widetilde{\delta}(\varsigma_n))^{-2}\prod_{\substack{k=1\\k\neq n}}^{m-1}\dfrac{d_k^-(\zeta)}{d_k^+(\zeta)} & (\zeta-\varsigma_n)\prod_{\substack{k=1\\k\neq n}}^{m-1}\dfrac{d_k^-(\zeta)}{d_k^+(\zeta)} \end{pmatrix},$$

$$\widetilde{J}_{\widetilde{\mathcal{L}}_n}(\zeta)=\begin{pmatrix} (\zeta-\overline{\varsigma_n})\prod_{\substack{k=1\\k\neq n}}^{m-1}\dfrac{d_k^+(\zeta)}{d_k^-(\zeta)} & \overline{g_n(\widetilde{\delta}(\varsigma_n))^{-2}}\prod_{\substack{k=1\\k\neq n}}^{m-1}\dfrac{d_k^+(\zeta)}{d_k^-(\zeta)} \\ -\dfrac{\widetilde{C}_n^{\mathcal{L}}}{(\zeta-\varsigma_n)^2}\prod_{\substack{k=1\\k\neq n}}^{m-1}\dfrac{d_k^-(\zeta)}{(d_k^+(\zeta))^{-1}} & \dfrac{\prod_{\substack{k=1\\k\neq n}}^{m-1}\frac{d_k^-(\zeta)}{d_k^+(\zeta)}-\frac{\widetilde{C}_n^{\mathcal{L}}\overline{g_n(\widetilde{\delta}(\varsigma_n))^{-2}}}{(\zeta-\varsigma_n)^2}\prod_{\substack{k=1\\k\neq n}}^{m-1}\frac{d_k^-(\zeta)}{(d_k^+(\zeta))^{-1}}}{(\zeta-\overline{\varsigma_n})} \end{pmatrix},$$

and

$$\widetilde{C}_n^{\mathcal{K}}=\overline{\widetilde{C}_n^{\mathcal{L}}}=-4\sin^2(\phi_n)(g_n)^{-1}(\widetilde{\delta}(\varsigma_n))^2\,\mathrm{e}^{-2\mathrm{i}\sum_{\substack{j=1\\j\neq n}}^{m-1}\phi_j}\prod_{\substack{k=1\\k\neq n}}^{m-1}\left(\frac{\sin(\frac{1}{2}(\phi_n+\phi_k))}{\sin(\frac{1}{2}(\phi_n-\phi_k))}\right)^2.$$

Then $\widetilde{m}^{\sharp}(\zeta)\colon \mathbb{C}\setminus\sigma_{\mathcal{OD}}''\to\mathrm{M}_2(\mathbb{C})$ *solves the following (augmented)* RHP:

(i) $\widetilde{m}^{\sharp}(\zeta)$ *is piecewise (sectionally) meromorphic* $\forall\,\zeta\in\mathbb{C}\setminus\sigma_c''$;

(ii) $\widetilde{m}^{\sharp}_{\pm}(\zeta):=\lim_{\substack{\zeta'\to\zeta\\ \zeta'\,\in\,\pm\,\text{side of }\sigma_{\mathcal{OD}}''}}\widetilde{m}^{\sharp}(\zeta')$ *satisfy the following jump conditions,*

$$\widetilde{m}^{\sharp}_{+}(\zeta)=\widetilde{m}^{\sharp}_{-}(\zeta)\exp(-\mathrm{i}k(\zeta)(x+2\lambda(\zeta)t)\mathrm{ad}(\sigma_3))\widetilde{\mathcal{G}}^{\sharp}(\zeta),\quad \zeta\in\mathbb{R},$$

where

$$\widetilde{\mathcal{G}}^{\sharp}(\zeta)=\begin{pmatrix} (1-r(\zeta)\overline{r(\overline{\zeta})})\widetilde{\delta}_-(\zeta)(\widetilde{\delta}_+(\zeta))^{-1} & -\overline{r(\overline{\zeta})}\,\widetilde{\delta}_-(\zeta)\widetilde{\delta}_+(\zeta)\prod_{k=1}^{m-1}(d_k^+(\zeta))^2 \\ r(\zeta)(\widetilde{\delta}_-(\zeta)\widetilde{\delta}_+(\zeta))^{-1}\prod_{k=1}^{m-1}(d_k^+(\zeta))^{-2} & (\widetilde{\delta}_-(\zeta))^{-1}\widetilde{\delta}_+(\zeta) \end{pmatrix},$$

and

$$\widetilde{m}^{\sharp}_{+}(\zeta)=\begin{cases}\widetilde{m}^{\sharp}_{-}(\zeta)\left(\mathrm{I}+\frac{\widetilde{C}_n^{\mathcal{K}}}{(\zeta-\varsigma_n)}\sigma_+\right), & \zeta\in\widetilde{\mathcal{K}}_n,\quad n\in\{1,2,\dots,m-1\},\\ \widetilde{m}^{\sharp}_{-}(\zeta)\left(\mathrm{I}+\frac{\widetilde{C}_n^{\mathcal{L}}}{(\zeta-\overline{\varsigma_n})}\sigma_-\right), & \zeta\in\widetilde{\mathcal{L}}_n,\quad n\in\{1,2,\dots,m-1\};\end{cases}$$

(iii) $\widetilde{m}^{\sharp}(\zeta)$ *has simple poles in* σ_d'' *with*

$$\begin{aligned}\mathrm{Res}(\widetilde{m}^{\sharp}(\zeta);\varsigma_n)&=\lim_{\zeta\to\varsigma_n}\widetilde{m}^{\sharp}(\zeta)g_n(\widetilde{\delta}(\varsigma_n))^{-2}\left(\prod_{k=1}^{m-1}(d_k^+(\varsigma_n))^{-2}\right)\sigma_-, && n\in\{m,m+1,\dots,N\},\\ \mathrm{Res}(\widetilde{m}^{\sharp}(\zeta);\overline{\varsigma_n})&=\sigma_1\overline{\mathrm{Res}(\widetilde{m}^{\sharp}(\zeta);\varsigma_n)}\,\sigma_1, && n\in\{m,m+1,\dots,N\};\end{aligned}$$

(iv) $\det(\widetilde{m}^{\sharp}(\zeta))|_{\zeta=\pm1}=0$;

(v) $\widetilde{m}^{\sharp}(\zeta)=_{\zeta\to0}\zeta^{-1}(\widetilde{\delta}(0))^{\sigma_3}\left(\prod_{k=1}^{m-1}(d_k^+(0))^{\sigma_3}\right)\sigma_2+\mathcal{O}(1)$;

(vi) $\widetilde{m}^{\sharp}(\zeta)\underset{\substack{\zeta\to\infty\\ \zeta\in\mathbb{C}\setminus\sigma_{\mathcal{OD}}''}}{=}\mathrm{I}+\mathcal{O}(\zeta^{-1})$;

(vii) $\widetilde{m}^{\sharp}(\zeta)=\sigma_1\overline{\widetilde{m}^{\sharp}(\overline{\zeta})}\,\sigma_1$ *and* $\widetilde{m}^{\sharp}(\zeta^{-1})=\zeta\widetilde{m}^{\sharp}(\zeta)(\widetilde{\delta}(0))^{\sigma_3}\left(\prod_{k=1}^{m-1}(d_k^+(0))^{\sigma_3}\right)\sigma_2$.

Let

$$u(x,t):=\mathrm{i}\lim_{\substack{\zeta\to\infty\\ \zeta\in\mathbb{C}\setminus\sigma_{\mathcal{OD}}''}}\left(\zeta\left(\widetilde{m}^{\sharp}(\zeta)(\widetilde{\delta}(\zeta))^{\sigma_3}\prod_{k=1}^{m-1}(d_k^+(\zeta))^{\sigma_3}-\mathrm{I}\right)\right)_{12},\tag{77}$$

and

$$\int_{+\infty}^{x}(|u(x',t)|^2-1)\,\mathrm{d}x':=-\mathrm{i}\lim_{\substack{\zeta\to\infty\\ \zeta\in\mathbb{C}\setminus\sigma_{\mathcal{OD}}''}}\left(\zeta\left(\widetilde{m}^{\sharp}(\zeta)(\widetilde{\delta}(\zeta))^{\sigma_3}\prod_{k=1}^{m-1}(d_k^+(\zeta))^{\sigma_3}-\mathrm{I}\right)\right)_{11}.\tag{78}$$

Then $u(x,t)$ *is the solution of the Cauchy problem for the* D_fNLSE.

The analogue of Proposition 3.1 is

Proposition A.1.1 ([38])**.** *The solution of the* RHP *for* $\widetilde{m}^{\sharp}(\zeta)\colon \mathbb{C}\setminus\sigma''_{\mathcal{O}\mathcal{D}}\to \mathrm{M}_2(\mathbb{C})$ *formulated in Lemma* A.1.3 *has the (integral equation) representation*

$$\widetilde{m}^{\sharp}(\zeta)=\left(\mathrm{I}+\zeta^{-1}\widetilde{\Delta}_o^{\sharp}\right)\widetilde{\mathcal{P}}^{\sharp}(\zeta)\left(\widetilde{m}_d^{\sharp}(\zeta)+\int_{\sigma''_c}\frac{\widetilde{m}_-^{\sharp}(\mu)(\widetilde{v}^{\sharp}(\mu)-\mathrm{I})}{(\mu-\zeta)}\frac{\mathrm{d}\mu}{2\pi\mathrm{i}}\right),\quad \zeta\in\mathbb{C}\setminus\sigma''_{\mathcal{O}\mathcal{D}},$$

where

$$\widetilde{m}_d^{\sharp}(\zeta)=\mathrm{I}+\sum_{n=m}^{N}\left(\frac{\mathrm{Res}(\widetilde{m}^{\sharp}(\zeta);\varsigma_n)}{(\zeta-\varsigma_n)}+\frac{\sigma_1\overline{\mathrm{Res}(\widetilde{m}^{\sharp}(\overline{\zeta});\varsigma_n)}\,\sigma_1}{(\zeta-\overline{\varsigma_n})}\right),$$

$\widetilde{v}^{\sharp}(\cdot)$ *is a generic notation for the jump matrices of* $\widetilde{m}^{\sharp}(\zeta)$ *on* σ''_c *(Lemma* A.1.3*, (ii)), and* $\widetilde{\Delta}_o^{\sharp}$ *and* $\widetilde{\mathcal{P}}^{\sharp}(\zeta)$ *are specified below. The solution of the above (integral) equation can be written as the ordered factorisation*

$$\widetilde{m}^{\sharp}(\zeta)=\left(\mathrm{I}+\zeta^{-1}\widetilde{\Delta}_o^{\sharp}\right)\widetilde{\mathcal{P}}^{\sharp}(\zeta)\widetilde{m}_d^{\sharp}(\zeta)\widetilde{m}^c(\zeta),\quad \zeta\in\mathbb{C}\setminus\sigma''_{\mathcal{O}\mathcal{D}},$$

where $\widetilde{m}_d^{\sharp}(\zeta)=\sigma_1\overline{\widetilde{m}_d^{\sharp}(\overline{\zeta})}\,\sigma_1$ $(\in\mathrm{SL}(2,\mathbb{C}))$ *has the representation given above,* $\widetilde{\mathcal{P}}^{\sharp}(\zeta)=\sigma_1\overline{\widetilde{\mathcal{P}}^{\sharp}(\overline{\zeta})}\,\sigma_1$ *is chosen so that* $\widetilde{\Delta}_o^{\sharp}$ *is idempotent,* $\mathrm{I}+\zeta^{-1}\widetilde{\Delta}_o^{\sharp}$ $(\in\mathrm{M}_2(\mathbb{C}))$ *is holomorphic in a punctured neighbourhood of the origin, with* $\widetilde{\Delta}_o^{\sharp}=\sigma_1\overline{\widetilde{\Delta}_o^{\sharp}}\,\sigma_1$ $(\in\mathrm{GL}(2,\mathbb{C}))$ *such that* $\det(\mathrm{I}+\zeta^{-1}\widetilde{\Delta}_o^{\sharp})|_{\zeta=\pm1}=0$*, and having the finite, order 2, matrix involutive structure* $\widetilde{\Delta}_o^{\sharp}=\begin{pmatrix}\widetilde{\Delta}^{\sharp}\mathrm{e}^{\mathrm{i}(k+1/2)\pi} & (1+(\widetilde{\Delta}^{\sharp})^2)^{1/2}\mathrm{e}^{-\mathrm{i}\widetilde{\vartheta}^{\sharp}}\\ (1+(\widetilde{\Delta}^{\sharp})^2)^{1/2}\mathrm{e}^{\mathrm{i}\widetilde{\vartheta}^{\sharp}} & \widetilde{\Delta}^{\sharp}\mathrm{e}^{-\mathrm{i}(k+1/2)\pi}\end{pmatrix}$*,* $k\in\mathbb{Z}$*, where* $\widetilde{\Delta}^{\sharp}$ *and* $\widetilde{\vartheta}^{\sharp}$ *are obtained from the relation* $\widetilde{\Delta}_o^{\sharp}=\widetilde{\mathcal{P}}^{\sharp}(0)\widetilde{m}_d^{\sharp}(0)\widetilde{m}^c(0)(\widetilde{\delta}(0))^{\sigma_3}\left(\prod_{k=1}^{m-1}(d_k^+(0))^{\sigma_3}\right)\sigma_2$*, and satisfying* $\mathrm{tr}(\widetilde{\Delta}_o^{\sharp})=0$*,* $\det(\widetilde{\Delta}_o^{\sharp})=-1$*, and* $\widetilde{\Delta}_o^{\sharp}\widetilde{\Delta}_o^{\sharp}=\mathrm{I}$*, and* $\widetilde{m}^c(\zeta)\colon\mathbb{C}\setminus\sigma''_c\to\mathrm{SL}(2,\mathbb{C})$ *solves the following* RHP: (1) $\widetilde{m}^c(\zeta)$ *is piecewise (sectionally) holomorphic* $\forall\,\zeta\in\mathbb{C}\setminus\sigma''_c$; (2) $\widetilde{m}_{\pm}^c(\zeta):=\lim_{\substack{\zeta'\to\zeta\\ \zeta'\in\,\pm\,\text{side of }\sigma''_c}}\widetilde{m}^c(\zeta')$ *satisfy the jump condition* $\widetilde{m}_+^c(\zeta)=\widetilde{m}_-^c(\zeta)\widetilde{v}^c(\zeta)$*,* $\zeta\in\sigma''_c$*, where* $\widetilde{v}^c(\zeta)=\exp(-\mathrm{i}k(\zeta)(x+2\lambda(\zeta)t)\mathrm{ad}(\sigma_3))\widetilde{\mathcal{G}}^{\sharp}(\zeta)$*,* $\zeta\in\mathbb{R}$*, with* $\widetilde{\mathcal{G}}^{\sharp}(\zeta)$ *given in Lemma* A.1.3*, (ii),* $\widetilde{v}^c(\zeta)=\mathrm{I}+\widetilde{C}_n^{\mathcal{K}}(\zeta-\varsigma_n)^{-1}\sigma_+$*,* $\zeta\in\widetilde{\mathcal{K}}_n$*, and* $\widetilde{v}^c(\zeta)=\mathrm{I}+\widetilde{C}_n^{\mathcal{L}}(\zeta-\overline{\varsigma_n})^{-1}\sigma_-$*,* $\zeta\in\widetilde{\mathcal{L}}_n$*,* $n\in\{1,2,\dots,m-1\}$*, with* $\widetilde{C}_n^{\mathcal{K}}$ *and* $\widetilde{C}_n^{\mathcal{L}}$ *given in Lemma* A.1.3; (3) $\widetilde{m}^c(\zeta)\underset{\substack{\zeta\to\infty\\ \zeta\in\mathbb{C}\setminus\sigma''_c}}{=}\mathrm{I}+\mathcal{O}(\zeta^{-1})$; *and* (4) $\widetilde{m}^c(\zeta)=\sigma_1\overline{\widetilde{m}^c(\overline{\zeta})}\,\sigma_1$.

The analogue of Lemma 3.5 is

Lemma A.1.4. *For* $m\in\{1,2,\dots,N\}$*, set* $\widetilde{\sigma}_d:=\cup_{n=m}^{N}(\{\varsigma_n\}\cup\{\overline{\varsigma_n}\})$*, and let* $\sigma_c=\{\zeta;\,\mathrm{Im}(\zeta)=0\}$ *with orientation from* $-\infty$ *to* $+\infty$*. Let* $\mathcal{X}(\zeta)\colon\mathbb{C}\setminus(\widetilde{\sigma}_d\cup\sigma_c)\to\mathrm{M}_2(\mathbb{C})$ *solve the following* RHP:

(i) $\mathcal{X}(\zeta)$ *is piecewise (sectionally) meromorphic* $\forall\,\zeta\in\mathbb{C}\setminus\sigma_c$;

(ii) $\mathcal{X}_{\pm}(\zeta):=\lim_{\substack{\zeta'\to\zeta\\ \zeta'\in\,\pm\,\text{side of }\sigma_c}}\mathcal{X}(\zeta')$ *satisfy the jump condition*

$$\mathcal{X}_+(\zeta)=\mathcal{X}_-(\zeta)\exp(-\mathrm{i}k(\zeta)(x+2\lambda(\zeta)t)\mathrm{ad}(\sigma_3))\widetilde{\mathcal{G}}^{\sharp}(\zeta),\quad \zeta\in\mathbb{R};$$

(iii) $\mathcal{X}(\zeta)$ *has simple poles in* $\widetilde{\sigma}_d$ *with*

$$\begin{aligned}\mathrm{Res}(\mathcal{X}(\zeta);\varsigma_n)&=\lim_{\zeta\to\varsigma_n}\mathcal{X}(\zeta)g_n(\widetilde{\delta}(\varsigma_n))^{-2}\left(\prod_{k=1}^{m-1}(d_k^+(\varsigma_n))^{-2}\right)\sigma_-, && n\in\{m,m+1,\dots,N\},\\ \mathrm{Res}(\mathcal{X}(\zeta);\overline{\varsigma_n})&=\sigma_1\overline{\mathrm{Res}(\mathcal{X}(\zeta);\varsigma_n)}\,\sigma_1, && n\in\{m,m+1,\dots,N\};\end{aligned}$$

(iv) $\det(\mathcal{X}(\zeta))|_{\zeta=\pm1}=0$;

(v) $\mathcal{X}(\zeta)=_{\zeta\to0}\zeta^{-1}(\widetilde{\delta}(0))^{\sigma_3}\left(\prod_{k=1}^{m-1}(d_k^+(0))^{\sigma_3}\right)\sigma_2+\mathcal{O}(1)$;

(vi) $\mathcal{X}(\zeta)\underset{\substack{\zeta\to\infty\\ \zeta\in\mathbb{C}\setminus(\widetilde{\sigma}_d\cup\sigma_c)}}{=}\mathrm{I}+\mathcal{O}(\zeta^{-1})$;

(vii) $\mathcal{X}(\zeta)=\sigma_1\overline{\mathcal{X}(\overline{\zeta})}\,\sigma_1$ *and* $\mathcal{X}(\zeta^{-1})=\zeta\mathcal{X}(\zeta)(\widetilde{\delta}(0))^{\sigma_3}\left(\prod_{k=1}^{m-1}(d_k^+(0))^{\sigma_3}\right)\sigma_2$.

Then, as $t\to-\infty$ *and* $x\to+\infty$ *such that* $z_o:=x/t<-2$ *and* $(x,t)\in\urcorner_m$, $\widetilde{m}^{\sharp}(\zeta)\colon\mathbb{C}\setminus\sigma''_{\mathcal{O}^{\mathcal{D}}}\to\mathrm{M}_2(\mathbb{C})$ *has the following asymptotics:*

$$\widetilde{m}^{\sharp}(\zeta)=\Big(\mathrm{I}+\mathcal{O}\Big(\widetilde{\mathfrak{F}}(\zeta)\exp\Big(-\widetilde{\beth}|t|\Big)\Big)\Big)\mathcal{X}(\zeta),$$

where $\widetilde{\beth}:=4\min_{\substack{m\in\{1,2,\dots,N\}\\ n\in\{1,2,\dots,m-1\}}}\{\sin(\phi_n)|\cos(\phi_n)-\cos(\phi_m)|\}$ (>0), *and, for* $i,j\in\{1,2\}$, $(\widetilde{\mathfrak{F}}(\zeta))_{ij}=_{\zeta\to\infty}\mathcal{O}(|\zeta|^{-1})$ *and* $(\widetilde{\mathfrak{F}}(\zeta))_{ij}=_{\zeta\to0}\mathcal{O}(1)$. *Furthermore, let*

$$u(x,t):=\mathrm{i}\lim_{\substack{\zeta\to\infty\\ \zeta\in\mathbb{C}\setminus(\widetilde{\sigma}_d\cup\sigma_c)}}\left(\zeta\left(\mathcal{X}(\zeta)(\widetilde{\delta}(\zeta))^{\sigma_3}\prod_{k=1}^{m-1}(d_k^+(\zeta))^{\sigma_3}-\mathrm{I}\right)\right)_{12}+\mathcal{O}\Big(\exp\Big(-\widetilde{\beth}|t|\Big)\Big),\tag{79}$$

and

$$\int_{+\infty}^{x}(|u(x',t)|^2-1)\,\mathrm{d}x':=-\mathrm{i}\lim_{\substack{\zeta\to\infty\\ \zeta\in\mathbb{C}\setminus(\widetilde{\sigma}_d\cup\sigma_c)}}\left(\zeta\left(\mathcal{X}(\zeta)(\widetilde{\delta}(\zeta))^{\sigma_3}\prod_{k=1}^{m-1}(d_k^+(\zeta))^{\sigma_3}-\mathrm{I}\right)\right)_{11}+\mathcal{O}\Big(\exp\Big(-\widetilde{\beth}|t|\Big)\Big).\tag{80}$$

Then $u(x,t)$ *is the solution of the Cauchy problem for the* D_fNLSE.

The analogue of Lemma 4.1 is

Lemma A.1.5. *The solution of the* RHP *for* $\mathcal{X}(\zeta)\colon\mathbb{C}\setminus(\widetilde{\sigma}_d\cup\sigma_c)\to\mathrm{M}_2(\mathbb{C})$ *formulated in Lemma* A.1.4 *is given by the following ordered factorisation,*

$$\mathcal{X}(\zeta)=\Big(\mathrm{I}+\zeta^{-1}\widetilde{\Delta}_o\Big)\widetilde{\mathcal{P}}(\zeta)\widetilde{m}_d(\zeta)\mathcal{M}^c(\zeta),\quad \zeta\in\mathbb{C}\setminus(\widetilde{\sigma}_d\cup\sigma_c),$$

where $\widetilde{m}_d(\zeta)=\sigma_1\overline{\widetilde{m}_d(\overline{\zeta})}\,\sigma_1$ $(\in\mathrm{SL}(2,\mathbb{C}))$ *has the (series) representation* $\widetilde{m}_d(\zeta)=\mathrm{I}+\sum_{n=m}^{N}(\frac{\mathrm{Res}(\mathcal{X}(\zeta);\varsigma_n)}{\zeta-\varsigma_n}+\frac{\sigma_1\overline{\mathrm{Res}(\mathcal{X}(\zeta);\varsigma_n)}\sigma_1}{\zeta-\overline{\varsigma_n}})$, $\widetilde{\mathcal{P}}(\zeta)=\sigma_1\overline{\widetilde{\mathcal{P}}(\overline{\zeta})}\,\sigma_1$ *is chosen (see Lemma* A.1.7 *below) so that* $\widetilde{\Delta}_o$ *is idempotent,* $\mathrm{I}+\zeta^{-1}\widetilde{\Delta}_o$ *is holomorphic in a punctured neighbourhood of the origin, with* $\widetilde{\Delta}_o=\sigma_1\overline{\widetilde{\Delta}_o}\,\sigma_1$ $(\in\mathrm{GL}(2,\mathbb{C}))$ *and* $\det(\mathrm{I}+\zeta^{-1}\widetilde{\Delta}_o)|_{\zeta=\pm1}=0$, *and determined by* $\widetilde{\Delta}_o=\widetilde{\mathcal{P}}(0)\widetilde{m}_d(0)\mathcal{M}^c(0)(\widetilde{\delta}(0))^{\sigma_3}\Big(\prod_{k=1}^{m-1}(d_k^+(0))^{\sigma_3}\Big)\sigma_2$, *and satisfying* $\mathrm{tr}(\widetilde{\Delta}_o)=0$, $\det(\widetilde{\Delta}_o)=-1$, *and* $\widetilde{\Delta}_o\widetilde{\Delta}_o=\mathrm{I}$, *and* $\mathcal{M}^c(\zeta)\colon\mathbb{C}\setminus\sigma_c\to\mathrm{SL}(2,\mathbb{C})$ *solves the following* RHP: (1) $\mathcal{M}^c(\zeta)$ *is piecewise (sectionally) holomorphic* $\forall\,\zeta\in\mathbb{C}\setminus\sigma_c$; (2) $\mathcal{M}^c_\pm(\zeta):=\lim_{\substack{\zeta'\to\zeta\\ \pm\mathrm{Im}(\zeta')>0}}\mathcal{M}^c(\zeta')$ *satisfy, for* $\zeta\in\mathbb{R}$, *the jump condition*

$$\mathcal{M}^c_+(\zeta)=\mathcal{M}^c_-(\zeta)\mathrm{e}^{-\mathrm{i}k(\zeta)(x+2\lambda(\zeta)t)\mathrm{ad}(\sigma_3)}\begin{pmatrix}(1-r(\zeta)\overline{r(\overline{\zeta})})\widetilde{\delta}_-(\zeta)/\widetilde{\delta}_+(\zeta) & -\frac{\overline{r(\overline{\zeta})}}{(\widetilde{\delta}_-(\zeta)\widetilde{\delta}_+(\zeta))^{-1}}\prod_{k=1}^{m-1}(d_k^+(\zeta))^2\\ \frac{r(\zeta)}{\widetilde{\delta}_-(\zeta)\widetilde{\delta}_+(\zeta)}\prod_{k=1}^{m-1}(d_k^+(\zeta))^{-2} & \widetilde{\delta}_+(\zeta)/\widetilde{\delta}_-(\zeta)\end{pmatrix};$$

(3) $\mathcal{M}^c(\zeta)=_{\substack{\zeta\to\infty\\ \zeta\in\mathbb{C}\setminus\sigma_c}}\mathrm{I}+\mathcal{O}(\zeta^{-1})$; *and* (4) $\mathcal{M}^c(\zeta)=\sigma_1\overline{\mathcal{M}^c(\overline{\zeta})}\,\sigma_1$.

The analogue of Lemma 4.2 is

Lemma A.1.6. *Let* ε *be an arbitrarily fixed, sufficiently small positive real number, and, for* $z\in\{\lambda_1,\lambda_2\}$, *with* λ_1 *and* λ_2 *given in Theorem* 2.2.1, *Eq.* (10), *set* $\mathbb{U}(z;\varepsilon):=\{\zeta;\,|\zeta-z|<\varepsilon\}$. *Then, as* $t\to-\infty$ *and* $x\to+\infty$ *such that* $z_o:=x/t<-2$, *for* $\zeta\in\mathbb{C}\setminus\cup_{z\in\{\lambda_1,\lambda_2\}}\mathbb{U}(z;\varepsilon)$, $\mathcal{M}^c(\zeta)$ *has the following asymptotics:*

$$\mathcal{M}^c_{11}(\zeta)=1+\mathcal{O}\left(\left(\frac{c^{\mathcal{S}}(\lambda_1)\underline{c}(\lambda_2,\lambda_3,\overline{\lambda_3})}{\sqrt{\lambda_2(z_o^2+32)}\,(\zeta-\lambda_1)}+\frac{c^{\mathcal{S}}(\lambda_2)\underline{c}(\lambda_1,\lambda_3,\overline{\lambda_3})}{\sqrt{\lambda_1(z_o^2+32)}\,(\zeta-\lambda_2)}\right)\frac{\ln|t|}{(\lambda_1-\lambda_2)t}\right),$$

$$\begin{aligned}\mathcal{M}^c_{12}(\zeta)=\mathrm{e}^{\frac{\mathrm{i}\Xi^-(0)}{2}}&\left(\frac{\sqrt{\nu(\lambda_1)}\,\lambda_1^{-2\mathrm{i}\nu(\lambda_1)}}{\sqrt{|t|(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\left(\frac{\lambda_1\mathrm{e}^{\mathrm{i}(\Theta^-(z_o,t)-\frac{3\pi}{4})}}{(\zeta-\lambda_1)}+\frac{\lambda_2\mathrm{e}^{-\mathrm{i}(\Theta^-(z_o,t)-\frac{3\pi}{4})}}{(\zeta-\lambda_2)}\right)\right.\\ &\left.+\mathcal{O}\left(\left(\frac{c^{\mathcal{S}}(\lambda_1)\underline{c}(\lambda_2,\lambda_3,\overline{\lambda_3})}{\sqrt{\lambda_2(z_o^2+32)}\,(\zeta-\lambda_1)}+\frac{c^{\mathcal{S}}(\lambda_2)\underline{c}(\lambda_1,\lambda_3,\overline{\lambda_3})}{\sqrt{\lambda_1(z_o^2+32)}\,(\zeta-\lambda_2)}\right)\frac{\ln|t|}{(\lambda_1-\lambda_2)t}\right)\right),\end{aligned}$$

$$
\begin{aligned}
\mathcal{M}_{21}^{c}(\zeta)=&\,\mathrm{e}^{-\frac{\mathrm{i}\Xi^{-}(0)}{2}}\left(\frac{\sqrt{\nu(\lambda_1)}\,\lambda_1^{2\mathrm{i}\nu(\lambda_1)}}{\sqrt{|t|(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\left(\frac{\lambda_1\mathrm{e}^{-\mathrm{i}(\Theta^{-}(z_o,t)-\frac{3\pi}{4})}}{(\zeta-\lambda_1)}+\frac{\lambda_2\mathrm{e}^{\mathrm{i}(\Theta^{-}(z_o,t)-\frac{3\pi}{4})}}{(\zeta-\lambda_2)}\right)\right.\\
&\left.+\mathcal{O}\left(\left(\frac{c^{\mathcal{S}}(\lambda_1)\underline{c}(\lambda_2,\lambda_3,\overline{\lambda_3})}{\sqrt{\lambda_2(z_o^2+32)}\,(\zeta-\lambda_1)}+\frac{c^{\mathcal{S}}(\lambda_2)\underline{c}(\lambda_1,\lambda_3,\overline{\lambda_3})}{\sqrt{\lambda_1(z_o^2+32)}\,(\zeta-\lambda_2)}\right)\frac{\ln|t|}{(\lambda_1-\lambda_2)t}\right)\right),\\
\mathcal{M}_{22}^{c}(\zeta)=&\,1+\mathcal{O}\left(\left(\frac{c^{\mathcal{S}}(\lambda_1)\underline{c}(\lambda_2,\lambda_3,\overline{\lambda_3})}{\sqrt{\lambda_2(z_o^2+32)}\,(\zeta-\lambda_1)}+\frac{c^{\mathcal{S}}(\lambda_2)\underline{c}(\lambda_1,\lambda_3,\overline{\lambda_3})}{\sqrt{\lambda_1(z_o^2+32)}\,(\zeta-\lambda_2)}\right)\frac{\ln|t|}{(\lambda_1-\lambda_2)t}\right),
\end{aligned}
$$

where λ_3*,* $\nu(\cdot)$*,* $\Theta^{-}(z_o,t)$*, and* $\Xi^{-}(\cdot)$*, respectively, are given in Theorem* 2.2.1*, Eqs.* (10), (11), (17), *and* (19)*,* $||(\cdot-\lambda_k)^{-1}||_{\mathcal{L}^{\infty}(\mathbb{C}\setminus\cup_{z\in\{\lambda_1,\lambda_2\}}\mathbb{U}(z;\varepsilon))}<\infty$*,* $k\in\{1,2\}$*,* $\mathcal{M}^{c}(\zeta)=\sigma_1\overline{\mathcal{M}^{c}(\overline{\zeta})}\,\sigma_1$*, and* $(\mathcal{M}^{c}(0)\sigma_2)^2=\mathrm{I}$ $(+\,\mathcal{O}(t^{-1}\ln|t|))$*.*

Sketch of Proof. Proceeding as in the proof of Lemma 6.1 in [38] and particularising it to the case of the RHP for $\mathcal{M}^{c}(\zeta)$ stated in Lemma A.1.5, one arrives at

$$
\begin{aligned}
\mathcal{M}_{11}^{c}(\zeta)=&\,1+\frac{\widetilde{r}(\lambda_1)(\widetilde{\delta}_B^0)^{-2}\mathrm{e}^{-\frac{3\pi\nu}{2}}\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}}{2\pi\mathrm{i}(\zeta-\lambda_1)\widetilde{\beta}_{21}^{\widetilde{\Sigma}_B^0}\,\widetilde{\mathcal{X}}_B\sqrt{|t|}}\int_0^{+\infty}(\mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}\partial_z\mathbf{D}_{\mathrm{i}\nu}(z)+\tfrac{\mathrm{i}}{2}\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}z\mathbf{D}_{\mathrm{i}\nu}(z))z^{\mathrm{i}\nu}\mathrm{e}^{-\frac{z^2}{4}}\,\mathrm{d}z\\
&-\frac{\widetilde{r}(\lambda_1)(1-|\widetilde{r}(\lambda_1)|^2)^{-1}(\widetilde{\delta}_B^0)^{-2}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}}{2\pi\mathrm{i}(\zeta-\lambda_1)\widetilde{\beta}_{21}^{\widetilde{\Sigma}_B^0}\mathrm{e}^{-\frac{\pi\nu}{2}}\,\widetilde{\mathcal{X}}_B\sqrt{|t|}}\int_0^{+\infty}(\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\partial_z\mathbf{D}_{\mathrm{i}\nu}(z)+\tfrac{\mathrm{i}}{2}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}z\mathbf{D}_{\mathrm{i}\nu}(z))z^{\mathrm{i}\nu}\mathrm{e}^{-\frac{z^2}{4}}\,\mathrm{d}z\\
&+\frac{\overline{\widetilde{r}(\lambda_1)}(\widetilde{\delta}_A^0)^{-2}\mathrm{e}^{-\frac{\pi\nu}{2}}(-1)^{-\mathrm{i}\nu}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}}{2\pi\mathrm{i}(\zeta-\lambda_2)\widetilde{\beta}_{21}^{\widetilde{\Sigma}_A^0}\,\widetilde{\mathcal{X}}_A\sqrt{|t|}}\int_0^{+\infty}(\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}\partial_z\mathbf{D}_{-\mathrm{i}\nu}(z)-\tfrac{\mathrm{i}}{2}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}z\mathbf{D}_{-\mathrm{i}\nu}(z))z^{-\mathrm{i}\nu}\mathrm{e}^{-\frac{z^2}{4}}\,\mathrm{d}z\\
&-\frac{\overline{\widetilde{r}(\lambda_1)}(1-|\widetilde{r}(\lambda_1)|^2)^{-1}(\widetilde{\delta}_A^0)^{-2}\mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}}{2\pi\mathrm{i}(\zeta-\lambda_2)\widetilde{\beta}_{21}^{\widetilde{\Sigma}_A^0}\mathrm{e}^{\frac{\pi\nu}{2}}(-1)^{\mathrm{i}\nu}\widetilde{\mathcal{X}}_A\sqrt{|t|}}\int_0^{+\infty}(\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}\partial_z\mathbf{D}_{-\mathrm{i}\nu}(z)-\tfrac{\mathrm{i}}{2}\mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}z\mathbf{D}_{-\mathrm{i}\nu}(z))z^{-\mathrm{i}\nu}\mathrm{e}^{-\frac{z^2}{4}}\,\mathrm{d}z\\
&+\mathcal{O}\left(\left(\frac{c^{\mathcal{S}}(\lambda_1)\underline{c}(\lambda_2,\lambda_3,\overline{\lambda_3})(\widetilde{\delta}_B^0)^{-2}}{(\zeta-\lambda_1)|\lambda_1-\lambda_3|\sqrt{(\lambda_1-\lambda_2)}\,\widetilde{\mathcal{X}}_B}+\frac{c^{\mathcal{S}}(\lambda_2)\underline{c}(\lambda_1,\lambda_3,\overline{\lambda_3})(\widetilde{\delta}_A^0)^{-2}}{(\zeta-\lambda_2)|\lambda_2-\lambda_3|\sqrt{(\lambda_1-\lambda_2)}\,\widetilde{\mathcal{X}}_A}\right)\frac{\ln|t|}{t}\right),\\
\mathcal{M}_{12}^{c}(\zeta)=&\left(\frac{\overline{\widetilde{r}(\lambda_1)}(1-|\widetilde{r}(\lambda_1)|^2)^{-1}(\widetilde{\delta}_B^0)^{2}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}}{2\pi\mathrm{i}(\zeta-\lambda_1)\mathrm{e}^{-\frac{\pi\nu}{2}}\,\widetilde{\mathcal{X}}_B\sqrt{|t|}}-\frac{\overline{\widetilde{r}(\lambda_1)}(\widetilde{\delta}_B^0)^{2}\mathrm{e}^{-\frac{3\pi\nu}{2}}\mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}}{2\pi\mathrm{i}(\zeta-\lambda_1)\widetilde{\mathcal{X}}_B\sqrt{|t|}}\right)\int_0^{+\infty}\mathbf{D}_{-\mathrm{i}\nu}(z)z^{-\mathrm{i}\nu}\mathrm{e}^{-\frac{z^2}{4}}\,\mathrm{d}z\\
&+\left(\frac{\widetilde{r}(\lambda_1)(1-|\widetilde{r}(\lambda_1)|^2)^{-1}(\widetilde{\delta}_A^0)^{2}\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}}{2\pi\mathrm{i}(\zeta-\lambda_2)\mathrm{e}^{\frac{\pi\nu}{2}}(-1)^{-\mathrm{i}\nu}\widetilde{\mathcal{X}}_A\sqrt{|t|}}-\frac{\widetilde{r}(\lambda_1)(\widetilde{\delta}_A^0)^{2}\mathrm{e}^{-\frac{\pi\nu}{2}}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}}{2\pi\mathrm{i}(\zeta-\lambda_2)(-1)^{-\mathrm{i}\nu}\widetilde{\mathcal{X}}_A\sqrt{|t|}}\right)\int_0^{+\infty}\mathbf{D}_{\mathrm{i}\nu}(z)z^{\mathrm{i}\nu}\mathrm{e}^{-\frac{z^2}{4}}\,\mathrm{d}z\\
&+\mathcal{O}\left(\left(\frac{c^{\mathcal{S}}(\lambda_1)\underline{c}(\lambda_2,\lambda_3,\overline{\lambda_3})(\widetilde{\delta}_B^0)^{2}}{(\zeta-\lambda_1)|\lambda_1-\lambda_3|\sqrt{(\lambda_1-\lambda_2)}\,\widetilde{\mathcal{X}}_B}+\frac{c^{\mathcal{S}}(\lambda_2)\underline{c}(\lambda_1,\lambda_3,\overline{\lambda_3})(\widetilde{\delta}_A^0)^{2}}{(\zeta-\lambda_2)|\lambda_2-\lambda_3|\sqrt{(\lambda_1-\lambda_2)}\,\widetilde{\mathcal{X}}_A}\right)\frac{\ln|t|}{t}\right),\\
\mathcal{M}_{21}^{c}(\zeta)=&-\left(\frac{\widetilde{r}(\lambda_1)(1-|\widetilde{r}(\lambda_1)|^2)^{-1}(\widetilde{\delta}_B^0)^{-2}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}}{2\pi\mathrm{i}(\zeta-\lambda_1)\mathrm{e}^{-\frac{\pi\nu}{2}}\,\widetilde{\mathcal{X}}_B\sqrt{|t|}}-\frac{\widetilde{r}(\lambda_1)(\widetilde{\delta}_B^0)^{-2}\mathrm{e}^{-\frac{3\pi\nu}{2}}\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}}{2\pi\mathrm{i}(\zeta-\lambda_1)\widetilde{\mathcal{X}}_B\sqrt{|t|}}\right)\int_0^{+\infty}\mathbf{D}_{\mathrm{i}\nu}(z)z^{\mathrm{i}\nu}\mathrm{e}^{-\frac{z^2}{4}}\,\mathrm{d}z\\
&-\left(\frac{\overline{\widetilde{r}(\lambda_1)}(1-|\widetilde{r}(\lambda_1)|^2)^{-1}(\widetilde{\delta}_A^0)^{-2}\mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}}{2\pi\mathrm{i}(\zeta-\lambda_2)\mathrm{e}^{\frac{\pi\nu}{2}}(-1)^{\mathrm{i}\nu}\widetilde{\mathcal{X}}_A\sqrt{|t|}}-\frac{\overline{\widetilde{r}(\lambda_1)}(\widetilde{\delta}_A^0)^{-2}\mathrm{e}^{-\frac{\pi\nu}{2}}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}}{2\pi\mathrm{i}(\zeta-\lambda_2)(-1)^{\mathrm{i}\nu}\widetilde{\mathcal{X}}_A\sqrt{|t|}}\right)\int_0^{+\infty}\mathbf{D}_{-\mathrm{i}\nu}(z)z^{-\mathrm{i}\nu}\mathrm{e}^{-\frac{z^2}{4}}\,\mathrm{d}z\\
&+\mathcal{O}\left(\left(\frac{c^{\mathcal{S}}(\lambda_1)\underline{c}(\lambda_2,\lambda_3,\overline{\lambda_3})(\widetilde{\delta}_B^0)^{-2}}{(\zeta-\lambda_1)|\lambda_1-\lambda_3|\sqrt{(\lambda_1-\lambda_2)}\,\widetilde{\mathcal{X}}_B}+\frac{c^{\mathcal{S}}(\lambda_2)\underline{c}(\lambda_1,\lambda_3,\overline{\lambda_3})(\widetilde{\delta}_A^0)^{-2}}{(\zeta-\lambda_2)|\lambda_2-\lambda_3|\sqrt{(\lambda_1-\lambda_2)}\,\widetilde{\mathcal{X}}_A}\right)\frac{\ln|t|}{t}\right),\\
\mathcal{M}_{22}^{c}(\zeta)=&\,1-\frac{\overline{\widetilde{r}(\lambda_1)}(\widetilde{\delta}_B^0)^{2}\mathrm{e}^{-\frac{3\pi\nu}{2}}\mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}}{2\pi\mathrm{i}(\zeta-\lambda_1)\widetilde{\beta}_{12}^{\widetilde{\Sigma}_B^0}\,\widetilde{\mathcal{X}}_B\sqrt{|t|}}\int_0^{+\infty}(\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}\partial_z\mathbf{D}_{-\mathrm{i}\nu}(z)-\tfrac{\mathrm{i}}{2}\mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}z\mathbf{D}_{-\mathrm{i}\nu}(z))z^{-\mathrm{i}\nu}\mathrm{e}^{-\frac{z^2}{4}}\,\mathrm{d}z\\
&+\frac{\overline{\widetilde{r}(\lambda_1)}(1-|\widetilde{r}(\lambda_1)|^2)^{-1}(\widetilde{\delta}_B^0)^{2}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}}{2\pi\mathrm{i}(\zeta-\lambda_1)\widetilde{\beta}_{12}^{\widetilde{\Sigma}_B^0}\mathrm{e}^{-\frac{\pi\nu}{2}}\,\widetilde{\mathcal{X}}_B\sqrt{|t|}}\int_0^{+\infty}(\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}\partial_z\mathbf{D}_{-\mathrm{i}\nu}(z)-\tfrac{\mathrm{i}}{2}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}z\mathbf{D}_{-\mathrm{i}\nu}(z))z^{-\mathrm{i}\nu}\mathrm{e}^{-\frac{z^2}{4}}\,\mathrm{d}z\\
&-\frac{\widetilde{r}(\lambda_1)(\widetilde{\delta}_A^0)^{2}\mathrm{e}^{-\frac{\pi\nu}{2}}(-1)^{\mathrm{i}\nu}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}}{2\pi\mathrm{i}(\zeta-\lambda_2)\widetilde{\beta}_{12}^{\widetilde{\Sigma}_A^0}\,\widetilde{\mathcal{X}}_A\sqrt{|t|}}\int_0^{+\infty}(\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\partial_z\mathbf{D}_{\mathrm{i}\nu}(z)+\tfrac{\mathrm{i}}{2}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}z\mathbf{D}_{\mathrm{i}\nu}(z))z^{\mathrm{i}\nu}\mathrm{e}^{-\frac{z^2}{4}}\,\mathrm{d}z\\
&+\frac{\widetilde{r}(\lambda_1)(1-|\widetilde{r}(\lambda_1)|^2)^{-1}(\widetilde{\delta}_A^0)^{2}(-1)^{\mathrm{i}\nu}\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}}{2\pi\mathrm{i}(\zeta-\lambda_2)\widetilde{\beta}_{12}^{\widetilde{\Sigma}_A^0}\mathrm{e}^{\frac{\pi\nu}{2}}\,\widetilde{\mathcal{X}}_A\sqrt{|t|}}\int_0^{+\infty}(\mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}\partial_z\mathbf{D}_{\mathrm{i}\nu}(z)+\tfrac{\mathrm{i}}{2}\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}z\mathbf{D}_{\mathrm{i}\nu}(z))z^{\mathrm{i}\nu}\mathrm{e}^{-\frac{z^2}{4}}\,\mathrm{d}z\\
&+\mathcal{O}\left(\left(\frac{c^{\mathcal{S}}(\lambda_1)\underline{c}(\lambda_2,\lambda_3,\overline{\lambda_3})(\widetilde{\delta}_B^0)^{2}}{(\zeta-\lambda_1)|\lambda_1-\lambda_3|\sqrt{(\lambda_1-\lambda_2)}\,\widetilde{\mathcal{X}}_B}+\frac{c^{\mathcal{S}}(\lambda_2)\underline{c}(\lambda_1,\lambda_3,\overline{\lambda_3})(\widetilde{\delta}_A^0)^{2}}{(\zeta-\lambda_2)|\lambda_2-\lambda_3|\sqrt{(\lambda_1-\lambda_2)}\,\widetilde{\mathcal{X}}_A}\right)\frac{\ln|t|}{t}\right),
\end{aligned}
$$

where $\widetilde{r}(\zeta)=r(\zeta)\prod_{k=1}^{m-1}(d_k^+(\zeta))^{-2}$ $(|\widetilde{r}(\lambda_1)|=|r(\lambda_1)|)$, $\nu=\nu(\lambda_1)$,

$$\begin{aligned}
\widetilde{\delta}_B^0&=|\lambda_1-\lambda_3|^{\mathrm{i}\nu}\left(2|t|(\lambda_1-\lambda_2)^3\lambda_1^{-3}\right)^{\frac{\mathrm{i}\nu}{2}}\mathrm{e}^{\mathcal{Y}(\lambda_1)}\exp\left(-\tfrac{\mathrm{i}t}{2}(\lambda_1-\lambda_2)(z_o+\lambda_1+\lambda_2)\right),\\
\widetilde{\delta}_A^0&=|\lambda_2-\lambda_3|^{-\mathrm{i}\nu}\left(2|t|(\lambda_1-\lambda_2)^3\lambda_2^{-3}\right)^{-\frac{\mathrm{i}\nu}{2}}\mathrm{e}^{\mathcal{Y}(\lambda_2)}\exp\left(\tfrac{\mathrm{i}t}{2}(\lambda_1-\lambda_2)(z_o+\lambda_1+\lambda_2)\right),\\
\mathcal{Y}(\lambda_1)&=\frac{\mathrm{i}}{2\pi}\int_0^{\lambda_2}\ln|\mu-\lambda_1|\mathrm{d}\ln(1-|r(\mu)|^2)+\frac{\mathrm{i}}{2\pi}\int_{\lambda_1}^{+\infty}\ln|\mu-\lambda_1|\mathrm{d}\ln(1-|r(\mu)|^2),\\
\mathcal{Y}(\lambda_2)&=-\mathcal{Y}(\lambda_1)+\frac{\mathrm{i}}{2\pi}\int_0^{\lambda_2}\ln|\mu|\mathrm{d}\ln(1-|r(\mu)|^2)+\frac{\mathrm{i}}{2\pi}\int_{\lambda_1}^{+\infty}\ln|\mu|\mathrm{d}\ln(1-|r(\mu)|^2),
\end{aligned}$$

$$\widetilde{\mathcal{X}}_B=\mathcal{X}_B,\quad \widetilde{\mathcal{X}}_A=\mathcal{X}_A,\quad \widetilde{\beta}_{12}^{\widetilde{\Sigma}_{B^0}}=\overline{\widetilde{\beta}_{21}^{\widetilde{\Sigma}_{B^0}}}=\frac{\sqrt{2\pi}\,\mathrm{e}^{-\frac{\pi\nu}{2}}\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}}{\widetilde{r}(\lambda_1)\Gamma(\mathrm{i}\nu)},\qquad \widetilde{\beta}_{12}^{\widetilde{\Sigma}_{A^0}}=\overline{\widetilde{\beta}_{21}^{\widetilde{\Sigma}_{A^0}}}=\frac{\sqrt{2\pi}\,\mathrm{e}^{-\frac{\pi\nu}{2}}\mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}}{\overline{\widetilde{r}(\lambda_1)}\,\overline{\Gamma(\mathrm{i}\nu)}},$$

$\Gamma(\cdot)$ is the gamma function [51], and $\mathbf{D}_*(\cdot)$ is the parabolic cylinder function [51]. Proceeding, now, as at the end of the sketch of the proof of Lemma 4.2, one obtains the result stated in the Lemma. Furthermore, one shows that the symmetry reduction $\mathcal{M}^c(\zeta)=\sigma_1\overline{\mathcal{M}^c(\overline{\zeta})}\,\sigma_1$ is satisfied, and verifies that $(\mathcal{M}^c(0)\sigma_2)^2=\mathrm{I}+\mathcal{O}(t^{-1}\ln|t|)$. $\square$

The analogue of Proposition 4.1 is

Proposition A.1.2. *For* $m\in\{1,2,\dots,N\}$, *set* $\mathrm{Res}(\mathcal{X}(\zeta);\varsigma_n):=\left(\begin{smallmatrix}\mathfrak{a}_n&\mathfrak{b}_n\\\mathfrak{c}_n&\mathfrak{d}_n\end{smallmatrix}\right)$, $n\in\{m,m+1,\dots,N\}$. *Then* $\mathfrak{b}_n=-\mathfrak{a}_n\mathcal{M}_{12}^c(\varsigma_n)/\mathcal{M}_{22}^c(\varsigma_n)$, $\mathfrak{d}_n=-\mathfrak{c}_n\mathcal{M}_{12}^c(\varsigma_n)/\mathcal{M}_{22}^c(\varsigma_n)$, *and* $\{\mathfrak{a}_n,\overline{\mathfrak{c}_n}\}_{n=m}^N$ *satisfy the following (non-singular) system of* $2(N-m+1)$ *linear inhomogeneous algebraic equations,*

$$\begin{bmatrix}\boxed{\widetilde{\mathcal{A}}}&\boxed{\widetilde{\mathcal{B}}}\\\boxed{\overline{\widetilde{\mathcal{B}}}}&\boxed{\overline{\widetilde{\mathcal{A}}}}\end{bmatrix}\begin{bmatrix}\mathfrak{a}_m\\\mathfrak{a}_{m+1}\\\vdots\\\mathfrak{a}_N\\\overline{\mathfrak{c}_m}\\\overline{\mathfrak{c}_{m+1}}\\\vdots\\\overline{\mathfrak{c}_N}\end{bmatrix}=\begin{bmatrix}g_m^\star\mathcal{M}_{12}^c(\varsigma_m)\\g_{m+1}^\star\mathcal{M}_{12}^c(\varsigma_{m+1})\\\vdots\\g_N^\star\mathcal{M}_{12}^c(\varsigma_N)\\\overline{g_m^\star\mathcal{M}_{22}^c(\varsigma_m)}\\\overline{g_{m+1}^\star\mathcal{M}_{22}^c(\varsigma_{m+1})}\\\vdots\\\overline{g_N^\star\mathcal{M}_{22}^c(\varsigma_N)}\end{bmatrix},$$

where

$$\widetilde{\mathcal{A}}_{ij}:=\begin{cases}\dfrac{\det(\mathcal{M}^c(\varsigma_i))+g_i^\star\mathrm{W}(\mathcal{M}_{12}^c(\varsigma_i),\mathcal{M}_{22}^c(\varsigma_i))}{\mathcal{M}_{22}^c(\varsigma_i)}, & i=j\in\{m,m+1,\dots,N\},\\[2mm] -\dfrac{g_i^\star(\mathcal{M}_{12}^c(\varsigma_i)\mathcal{M}_{22}^c(\varsigma_j)-\mathcal{M}_{22}^c(\varsigma_i)\mathcal{M}_{12}^c(\varsigma_j))}{(\varsigma_i-\varsigma_j)\mathcal{M}_{22}^c(\varsigma_j)}, & i\neq j\in\{m,m+1,\dots,N\},\end{cases}$$

$$\widetilde{\mathcal{B}}_{ij}:=-\frac{g_i^\star(\mathcal{M}_{22}^c(\varsigma_i)\overline{\mathcal{M}_{22}^c(\varsigma_j)}-\mathcal{M}_{12}^c(\varsigma_i)\overline{\mathcal{M}_{12}^c(\varsigma_j)})}{(\varsigma_i-\overline{\varsigma_j})\overline{\mathcal{M}_{22}^c(\varsigma_j)}},\qquad i,j\in\{m,m+1,\dots,N\},$$

$$g_j^\star=|g_j|\mathrm{e}^{\mathrm{i}\theta_{g_j}}\exp(2\mathrm{i}k(\varsigma_j)(x+2\lambda(\varsigma_j)t))(\widetilde{\delta}(\varsigma_j))^{-2}\prod_{k=1}^{m-1}(d_k^+(\varsigma_j))^{-2},\quad j\in\{m,m+1,\dots,N\},$$

with $|g_j|$ *and* θ_{g_j} *given in Lemma* 3.1, *(iii), and* $\mathrm{W}(\mathcal{M}_{12}^c(z),\mathcal{M}_{22}^c(z))=\left|\begin{smallmatrix}\mathcal{M}_{12}^c(z)&\mathcal{M}_{22}^c(z)\\\partial_z\mathcal{M}_{12}^c(z)&\partial_z\mathcal{M}_{22}^c(z)\end{smallmatrix}\right|$.

The analogue of Proposition 4.2 is

Proposition A.1.3. *As* $t\to-\infty$ *and* $x\to+\infty$ *such that* $z_o:=x/t<-2$ *and* $(x,t)\in\urcorner_m$, $m\in\{1,2,\dots,m\}$, *for* $n\in\{m+1,m+2,\dots,N\}$,

$$\begin{aligned}
\mathfrak{a}_n&=\mathcal{O}\left(\mathrm{e}^{-\beth^-|t|}\right),&\qquad \mathfrak{b}_n&=\mathcal{O}\left(t^{-1/2}(z_o^2+32)^{-1/4}\mathrm{e}^{-\beth^-|t|}\right),\\
\mathfrak{c}_n&=\mathcal{O}\left(\mathrm{e}^{-\beth^-|t|}\right),&\qquad \mathfrak{d}_n&=\mathcal{O}\left(t^{-1/2}(z_o^2+32)^{-1/4}\mathrm{e}^{-\beth^-|t|}\right),
\end{aligned}$$

where $\daleth^{-} := 4 \min_{\substack{m\in\{1,2,\dots,N\} \\ n\in\{m+1,m+2,\dots,N\}}} \{\sin(\phi_n)|\cos(\phi_n)-\cos(\phi_m)|\}$ (>0), *and*

$$
\begin{aligned}
\mathfrak{a}_m =& \; \mathfrak{a}_m^0 + \frac{1}{\sqrt{|t|}}\mathfrak{a}_m^1 + \mathcal{O}\left(\frac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\frac{\ln|t|}{t}\right) \\
=:& \; \frac{g_m^{\star}\overline{g_m^{\star}}(\varsigma_m-\overline{\varsigma_m})^{-1}}{(1+g_m^{\star}\overline{g_m^{\star}}(\varsigma_m-\overline{\varsigma_m})^{-2})} + \frac{1}{\sqrt{|t|}}\left(\frac{g_m^{\star}\overline{g_m^{\star}}(\varsigma_m-\overline{\varsigma_m})^{-1}(g_m^{\star}\partial_{\zeta}\widetilde{\mathcal{M}}_{12}^c(\varsigma_m)+\overline{g_m^{\star}\partial_{\zeta}\widetilde{\mathcal{M}}_{12}^c(\varsigma_m)})}{(1+g_m^{\star}\overline{g_m^{\star}}(\varsigma_m-\overline{\varsigma_m})^{-2})^2} + \frac{g_m^{\star}\widetilde{\mathcal{M}}_{12}^c(\varsigma_m)}{(1+g_m^{\star}\overline{g_m^{\star}}(\varsigma_m-\overline{\varsigma_m})^{-2})}\right) \\
&+ \mathcal{O}\left(\frac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\frac{\ln|t|}{t}\right), \\
\mathfrak{b}_m =& \; \frac{1}{\sqrt{|t|}}\mathfrak{b}_m^1 + \mathcal{O}\left(\frac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\frac{\ln|t|}{t}\right) =: -\frac{1}{\sqrt{|t|}}\frac{g_m^{\star}\overline{g_m^{\star}}(\varsigma_m-\overline{\varsigma_m})^{-1}\widetilde{\mathcal{M}}_{12}^c(\varsigma_m)}{(1+g_m^{\star}\overline{g_m^{\star}}(\varsigma_m-\overline{\varsigma_m})^{-2})} + \mathcal{O}\left(\frac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\frac{\ln|t|}{t}\right), \\
\mathfrak{c}_m =& \; \mathfrak{c}_m^0 + \frac{1}{\sqrt{|t|}}\mathfrak{c}_m^1 + \mathcal{O}\left(\frac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\frac{\ln|t|}{t}\right) \\
=:& \; \frac{g_m^{\star}}{(1+g_m^{\star}\overline{g_m^{\star}}(\varsigma_m-\overline{\varsigma_m})^{-2})} + \frac{1}{\sqrt{|t|}}\left(\frac{g_m^{\star}\overline{g_m^{\star}}(\varsigma_m-\overline{\varsigma_m})^{-1}\overline{\widetilde{\mathcal{M}}_{12}^c(\varsigma_m)} - g_m^{\star}\overline{g_m^{\star}\partial_{\zeta}\widetilde{\mathcal{M}}_{12}^c(\varsigma_m)}}{(1+g_m^{\star}\overline{g_m^{\star}}(\varsigma_m-\overline{\varsigma_m})^{-2})} + \frac{g_m^{\star}(g_m^{\star}\partial_{\zeta}\widetilde{\mathcal{M}}_{12}^c(\varsigma_m)+\overline{g_m^{\star}\partial_{\zeta}\widetilde{\mathcal{M}}_{12}^c(\varsigma_m)})}{(1+g_m^{\star}\overline{g_m^{\star}}(\varsigma_m-\overline{\varsigma_m})^{-2})^2}\right) \\
&+ \mathcal{O}\left(\frac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\frac{\ln|t|}{t}\right), \\
\mathfrak{d}_m =& \; \frac{1}{\sqrt{|t|}}\mathfrak{d}_m^1 + \mathcal{O}\left(\frac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\frac{\ln|t|}{t}\right) =: -\frac{1}{\sqrt{|t|}}\frac{g_m^{\star}\widetilde{\mathcal{M}}_{12}^c(\varsigma_m)}{(1+g_m^{\star}\overline{g_m^{\star}}(\varsigma_m-\overline{\varsigma_m})^{-2})} + \mathcal{O}\left(\frac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\frac{\ln|t|}{t}\right),
\end{aligned}
$$

where

$$
\widetilde{\mathcal{M}}_{12}^c(\zeta) = \frac{\sqrt{\nu(\lambda_1)}\,\mathrm{e}^{\frac{\mathrm{i}\Xi^{-}(0)}{2}}\lambda_1^{-2\mathrm{i}\nu(\lambda_1)}}{\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\left(\frac{\lambda_1\mathrm{e}^{\mathrm{i}(\Theta^{-}(z_o,t)-\frac{3\pi}{4})}}{(\zeta-\lambda_1)} + \frac{\lambda_2\mathrm{e}^{-\mathrm{i}(\Theta^{-}(z_o,t)-\frac{3\pi}{4})}}{(\zeta-\lambda_2)}\right),
$$

with $\nu(\cdot)$, λ_1, λ_2, λ_3, $\Xi^{-}(\cdot)$, *and* $\Theta^{-}(z_o,t)$ *specified in Lemma* A.1.6, *and* $c^{\mathcal{S}}(z_o)$ *given in Proposition* 4.2. *Furthermore, setting* $\widetilde{\mathcal{Y}} := \begin{pmatrix} \boxed{\widetilde{\mathcal{A}}} & \boxed{\widetilde{\mathcal{B}}} \\ \boxed{\overline{\widetilde{\mathcal{B}}}} & \boxed{\overline{\widetilde{\mathcal{A}}}} \end{pmatrix}$,

$$
0 < |\det(\widetilde{\mathcal{Y}})|^2 \leqslant \prod_{j=m}^{N}\left(1+\frac{\sin^2(\phi_m)|\gamma_m|^2P^{-2}(\phi_m,\phi_k)Q^{-2}(\phi_m)}{\sin^2(\frac{1}{2}(\phi_m+\phi_j))}\mathrm{e}^{2\phi(x,t)}\right)^2 + \mathcal{O}\left(\frac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\frac{\ln|t|}{t}\right), \qquad (81)
$$

where $\phi(x,t)$, $P(\phi_m,\phi_k)$, *and* $Q(\phi_m)$ *are defined in Eqs.* (67), (68), *and* (69), *respectively.*

The analogue of Lemma 4.3 is (see, also, Remark 4.1)

Lemma A.1.7. *As* $t\to-\infty$ *and* $x\to+\infty$ *such that* $z_o := x/t < -2$ *and* $(x,t)\in\urcorner_m$, $m\in\{1,2,\dots,N\}$,

$$
\widetilde{\mathcal{P}}(\zeta) = \begin{pmatrix} \frac{\zeta+\widetilde{a}_1^{-}}{\zeta+\widetilde{a}_2^{-}} & \frac{\widetilde{a}_3^{-}}{\zeta+\widetilde{a}_4^{-}} \\ \frac{\overline{\widetilde{a}_3^{-}}}{\zeta+\overline{\widetilde{a}_4^{-}}} & \frac{\zeta+\overline{\widetilde{a}_1^{-}}}{\zeta+\overline{\widetilde{a}_2^{-}}} \end{pmatrix},
$$

where

$$
\begin{aligned}
\widetilde{a}_1^{-} &= \overline{\widetilde{a}_2^{-}} = 1+\sum_{p=1}^{\infty}\sum_{q=0}^{p-1}\frac{\widetilde{a}_{pq}^1(z_o)(\ln|t|)^q}{|t|^{p/2}} + \mathcal{O}\left(\mathrm{e}^{-4|t|\min_{\substack{m\in\{1,2,\dots,N\} \\ n\in\{m+1,m+2,\dots,N\}}}\{\sin(\phi_n)|\cos(\phi_n)-\cos(\phi_m)|\}}\right), \\
\widetilde{a}_3^{-} &= \sum_{p=1}^{\infty}\sum_{q=0}^{p-1}\frac{\widetilde{a}_{pq}^3(z_o)(\ln|t|)^q}{|t|^{p/2}} + \mathcal{O}\left(\mathrm{e}^{-4|t|\min_{\substack{m\in\{1,2,\dots,N\} \\ n\in\{m+1,m+2,\dots,N\}}}\{\sin(\phi_n)|\cos(\phi_n)-\cos(\phi_m)|\}}\right), \\
\widetilde{a}_4^{-} &= 1+\sum_{p=1}^{\infty}\sum_{q=0}^{p-1}\frac{\widetilde{a}_{pq}^4(z_o)(\ln|t|)^q}{|t|^{p/2}} + \mathcal{O}\left(\mathrm{e}^{-4|t|\min_{\substack{m\in\{1,2,\dots,N\} \\ n\in\{m+1,m+2,\dots,N\}}}\{\sin(\phi_n)|\cos(\phi_n)-\cos(\phi_m)|\}}\right),
\end{aligned}
$$

$\widetilde{a}_{pq}^k(z_o)\in c^{\mathcal{S}}(z_o)$, $k\in\{1,3,4\}$, *and* $\widetilde{\mathcal{P}}(\zeta) = \sigma_1\overline{\widetilde{\mathcal{P}}(\overline{\zeta})}\,\sigma_1$.

The analogue of Proposition 4.3 is

Proposition A.1.4. *Set* $\widetilde{a}^{1}_{10}(z_o)=:\widetilde{a}_1$, $\widetilde{a}^{2}_{10}(z_o)=:\widetilde{a}_2$, $\widetilde{a}^{3}_{10}(z_o)=:\widetilde{a}_3$, *and* $\widetilde{a}^{4}_{10}(z_o)=:\widetilde{a}_4$. *Then as* $t\to-\infty$ *and* $x\to+\infty$ *such that* $z_o:=x/t<-2$ *and* $(x,t)\in\urcorner_m$, $m\in\{1,2,\dots,N\}$,

$$
\begin{aligned}
(\widetilde{\Delta}_o)_{11}= &-\frac{\overline{\mathfrak{c}^{0}_{m}}}{\overline{\varsigma_{m}}}\mathrm{i}\widetilde{\delta}^{-1}(0)\mathrm{e}^{2\mathrm{i}\sum_{k=1}^{m-1}\phi_k}+\frac{\mathrm{i}\widetilde{\delta}^{-1}(0)\mathrm{e}^{2\mathrm{i}\sum_{k=1}^{m-1}\phi_k}}{\sqrt{|t|}}\Bigg(-(\widetilde{a}_1-\widetilde{a}_2)\frac{\overline{\mathfrak{c}^{0}_{m}}}{\overline{\varsigma_{m}}}-\left(\frac{\mathfrak{b}^{1}_{m}}{\varsigma_m}+\frac{\overline{\mathfrak{c}^{1}_{m}}}{\overline{\varsigma_{m}}}\right)\\
&+\widetilde{a}_3\left(1-\frac{\overline{\mathfrak{a}^{0}_{m}}}{\overline{\varsigma_{m}}}\right)-\left(1-\frac{\mathfrak{a}^{0}_{m}}{\varsigma_m}\right)\frac{2\widetilde{\delta}(0)\sqrt{\nu(\lambda_1)}\,\cos(\Theta^{-}(z_o,t)-\frac{3\pi}{4})}{\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\Bigg)+\mathcal{O}\left(\frac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\frac{\ln|t|}{t}\right),\\
(\widetilde{\Delta}_o)_{12}= &-\left(1-\frac{\mathfrak{a}^{0}_{m}}{\varsigma_m}\right)\mathrm{i}\widetilde{\delta}(0)\mathrm{e}^{-2\mathrm{i}\sum_{k=1}^{m-1}\phi_k}+\frac{\mathrm{i}\widetilde{\delta}(0)\mathrm{e}^{-2\mathrm{i}\sum_{k=1}^{m-1}\phi_k}}{\sqrt{|t|}}\Bigg(-(\widetilde{a}_1-\widetilde{a}_2)\left(1-\frac{\mathfrak{a}^{0}_{m}}{\varsigma_m}\right)+\left(\frac{\mathfrak{a}^{1}_{m}}{\varsigma_m}+\frac{\overline{\mathfrak{d}^{1}_{m}}}{\overline{\varsigma_{m}}}\right)\\
&+\widetilde{a}_3\frac{\mathfrak{c}^{0}_{m}}{\varsigma_m}-\frac{\overline{\mathfrak{c}^{0}_{m}}}{\overline{\varsigma_{m}}}\frac{2\widetilde{\delta}^{-1}(0)\sqrt{\nu(\lambda_1)}\,\cos(\Theta^{-}(z_o,t)-\frac{3\pi}{4})}{\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\Bigg)+\mathcal{O}\left(\frac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\frac{\ln|t|}{t}\right),\\
(\widetilde{\Delta}_o)_{21}= &\left(1-\frac{\overline{\mathfrak{a}^{0}_{m}}}{\overline{\varsigma_{m}}}\right)\mathrm{i}\widetilde{\delta}^{-1}(0)\mathrm{e}^{2\mathrm{i}\sum_{k=1}^{m-1}\phi_k}+\frac{\mathrm{i}\widetilde{\delta}^{-1}(0)\mathrm{e}^{2\mathrm{i}\sum_{k=1}^{m-1}\phi_k}}{\sqrt{|t|}}\Bigg((\overline{\widetilde{a}_1}-\overline{\widetilde{a}_2})\left(1-\frac{\overline{\mathfrak{a}^{0}_{m}}}{\overline{\varsigma_{m}}}\right)-\left(\frac{\overline{\mathfrak{a}^{1}_{m}}}{\overline{\varsigma_{m}}}+\frac{\mathfrak{d}^{1}_{m}}{\varsigma_m}\right)\\
&-\overline{\widetilde{a}_3}\frac{\overline{\mathfrak{c}^{0}_{m}}}{\overline{\varsigma_{m}}}+\frac{\mathfrak{c}^{0}_{m}}{\varsigma_m}\frac{2\widetilde{\delta}(0)\sqrt{\nu(\lambda_1)}\,\cos(\Theta^{-}(z_o,t)-\frac{3\pi}{4})}{\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\Bigg)+\mathcal{O}\left(\frac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\frac{\ln|t|}{t}\right),\\
(\widetilde{\Delta}_o)_{22}= &\frac{\mathfrak{c}^{0}_{m}}{\varsigma_m}\mathrm{i}\widetilde{\delta}(0)\mathrm{e}^{-2\mathrm{i}\sum_{k=1}^{m-1}\phi_k}+\frac{\mathrm{i}\widetilde{\delta}(0)\mathrm{e}^{-2\mathrm{i}\sum_{k=1}^{m-1}\phi_k}}{\sqrt{|t|}}\Bigg((\overline{\widetilde{a}_1}-\overline{\widetilde{a}_2})\frac{\mathfrak{c}^{0}_{m}}{\varsigma_m}+\left(\frac{\overline{\mathfrak{b}^{1}_{m}}}{\overline{\varsigma_{m}}}+\frac{\mathfrak{c}^{1}_{m}}{\varsigma_m}\right)\\
&-\overline{\widetilde{a}_3}\left(1-\frac{\mathfrak{a}^{0}_{m}}{\varsigma_m}\right)+\left(1-\frac{\overline{\mathfrak{a}^{0}_{m}}}{\overline{\varsigma_{m}}}\right)\frac{2\widetilde{\delta}^{-1}(0)\sqrt{\nu(\lambda_1)}\,\cos(\Theta^{-}(z_o,t)-\frac{3\pi}{4})}{\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\Bigg)+\mathcal{O}\left(\frac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\frac{\ln|t|}{t}\right).
\end{aligned}
$$

The analogue of Proposition 4.4 is

Proposition A.1.5. *Let* $\phi(x,t)$, $P(\phi_m,\phi_k)$, *and* $Q(\phi_m)$ *be defined by Eqs.* (67), (68), *and* (69), *respectively. Then, for* $\theta_{\gamma_m}=\pm\pi/2$, *as* $t\to-\infty$ *and* $x\to+\infty$ *such that* $z_o:=x/t<-2$ *and* $(x,t)\in\urcorner_m$, $m\in\{1,2,\dots,N\}$,

$$
\begin{aligned}
\mathfrak{a}^{0}_{m}= &-\frac{2\mathrm{i}\sin(\phi_m)|\gamma_m|^2P^{-2}(\phi_m,\phi_k)Q^{-2}(\phi_m)\mathrm{e}^{2\phi(x,t)}}{(1-|\gamma_m|^2P^{-2}(\phi_m,\phi_k)Q^{-2}(\phi_m)\mathrm{e}^{2\phi(x,t)})},\\
\mathfrak{c}^{0}_{m}= &\mp\frac{2\sin(\phi_m)|\gamma_m|\widetilde{\delta}^{-1}(0)\mathrm{e}^{\mathrm{i}(\phi_m+s^{-})+\phi(x,t)}P^{-1}(\phi_m,\phi_k)Q^{-1}(\phi_m)}{(1-|\gamma_m|^2P^{-2}(\phi_m,\phi_k)Q^{-2}(\phi_m)\mathrm{e}^{2\phi(x,t)})},\\
\mathfrak{a}^{1}_{m}= &\mp\frac{16\mathrm{i}\lambda_1^2\sin^2(\phi_m)|\gamma_m|^3\sqrt{\nu(\lambda_1)}\,P^{-3}(\phi_m,\phi_k)Q^{-3}(\phi_m)\cos(s^{-})\mathrm{e}^{3\phi(x,t)}}{(1-|\gamma_m|^2P^{-2}(\phi_m,\phi_k)Q^{-2}(\phi_m)\mathrm{e}^{2\phi(x,t)})^2(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)^2\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\\
&\times\big(((\lambda_1+\lambda_2)\cos(\phi_m)-2)\cos(\Theta^{-}(z_o,t)-\tfrac{3\pi}{4})-(\lambda_1-\lambda_2)\sin(\phi_m)\sin(\Theta^{-}(z_o,t)-\tfrac{3\pi}{4})\big)\\
&\mp\frac{2\lambda_1\sin(\phi_m)|\gamma_m|\sqrt{\nu(\lambda_1)}\,P^{-1}(\phi_m,\phi_k)Q^{-1}(\phi_m)\mathrm{e}^{\phi(x,t)}}{(1-|\gamma_m|^2P^{-2}(\phi_m,\phi_k)Q^{-2}(\phi_m)\mathrm{e}^{2\phi(x,t)})(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\\
&\times\big(2\cos(s^{-})\cos(\Theta^{-}(z_o,t)-\tfrac{3\pi}{4})-(\lambda_1+\lambda_2)\cos(\phi_m+s^{-})\cos(\Theta^{-}(z_o,t)-\tfrac{3\pi}{4})\\
&+(\lambda_1-\lambda_2)\sin(\phi_m+s^{-})\sin(\Theta^{-}(z_o,t)-\tfrac{3\pi}{4})+2\mathrm{i}\sin(s^{-})\cos(\Theta^{-}(z_o,t)-\tfrac{3\pi}{4})\\
&-\mathrm{i}(\lambda_1+\lambda_2)\sin(\phi_m+s^{-})\cos(\Theta^{-}(z_o,t)-\tfrac{3\pi}{4})-\mathrm{i}(\lambda_1-\lambda_2)\cos(\phi_m+s^{-})\sin(\Theta^{-}(z_o,t)-\tfrac{3\pi}{4})\big),\\
\mathfrak{b}^{1}_{m}= &\frac{2\mathrm{i}\lambda_1\sin(\phi_m)|\gamma_m|^2\sqrt{\nu(\lambda_1)}\,\widetilde{\delta}(0)\mathrm{e}^{-\mathrm{i}(\phi_m+s^{-})+2\phi(x,t)}P^{-2}(\phi_m,\phi_k)Q^{-2}(\phi_m)}{(1-|\gamma_m|^2P^{-2}(\phi_m,\phi_k)Q^{-2}(\phi_m)\mathrm{e}^{2\phi(x,t)})(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\\
&\times\big(2\cos(s^{-})\cos(\Theta^{-}(z_o,t)-\tfrac{3\pi}{4})-(\lambda_1+\lambda_2)\cos(\phi_m+s^{-})\cos(\Theta^{-}(z_o,t)-\tfrac{3\pi}{4})\\
&+(\lambda_1-\lambda_2)\sin(\phi_m+s^{-})\sin(\Theta^{-}(z_o,t)-\tfrac{3\pi}{4})+2\mathrm{i}\sin(s^{-})\cos(\Theta^{-}(z_o,t)-\tfrac{3\pi}{4})\\
&-\mathrm{i}(\lambda_1+\lambda_2)\sin(\phi_m+s^{-})\cos(\Theta^{-}(z_o,t)-\tfrac{3\pi}{4})-\mathrm{i}(\lambda_1-\lambda_2)\cos(\phi_m+s^{-})\sin(\Theta^{-}(z_o,t)-\tfrac{3\pi}{4})\big),\\
\mathfrak{c}^{1}_{m}= &-\frac{16\lambda_1^2\sin^2(\phi_m)|\gamma_m|^2\sqrt{\nu(\lambda_1)}\,\widetilde{\delta}^{-1}(0)\mathrm{e}^{\mathrm{i}(\phi_m+s^{-})+2\phi(x,t)}P^{-2}(\phi_m,\phi_k)Q^{-2}(\phi_m)\cos(s^{-})}{(1-|\gamma_m|^2P^{-2}(\phi_m,\phi_k)Q^{-2}(\phi_m)\mathrm{e}^{2\phi(x,t)})^2(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)^2\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\\
&\times\big(((\lambda_1+\lambda_2)\cos(\phi_m)-2)\cos(\Theta^{-}(z_o,t)-\tfrac{3\pi}{4})-(\lambda_1-\lambda_2)\sin(\phi_m)\sin(\Theta^{-}(z_o,t)-\tfrac{3\pi}{4})\big)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{2\mathrm{i}\lambda_1\sin(\phi_m)|\gamma_m|^2\sqrt{\nu(\lambda_1)}\,\widetilde{\delta}^{-1}(0)\mathrm{e}^{\mathrm{i}(\phi_m+s^-)+2\phi(x,t)}P^{-2}(\phi_m,\phi_k)Q^{-2}(\phi_m)}{(1-|\gamma_m|^2P^{-2}(\phi_m,\phi_k)Q^{-2}(\phi_m)\mathrm{e}^{2\phi(x,t)})(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\\
&\times\big(2\cos(s^-)\cos(\Theta^-(z_o,t)-\tfrac{3\pi}{4})-(\lambda_1+\lambda_2)\cos(\phi_m+s^-)\cos(\Theta^-(z_o,t)-\tfrac{3\pi}{4})\\
&+(\lambda_1-\lambda_2)\sin(\phi_m+s^+)\sin(\Theta^-(z_o,t)-\tfrac{3\pi}{4})-2\mathrm{i}\sin(s^-)\cos(\Theta^-(z_o,t)-\tfrac{3\pi}{4})\\
&+\mathrm{i}(\lambda_1+\lambda_2)\sin(\phi_m+s^-)\cos(\Theta^-(z_o,t)-\tfrac{3\pi}{4})+\mathrm{i}(\lambda_1-\lambda_2)\cos(\phi_m+s^-)\sin(\Theta^-(z_o,t)-\tfrac{3\pi}{4})\big)\\
&+\frac{8\lambda_1^2\sin^2(\phi_m)|\gamma_m|^2\sqrt{\nu(\lambda_1)}\,\widetilde{\delta}^{-1}(0)\mathrm{e}^{\mathrm{i}(\phi_m+s^-)+2\phi(x,t)}P^{-2}(\phi_m,\phi_k)Q^{-2}(\phi_m)}{(1-|\gamma_m|^2P^{-2}(\phi_m,\phi_k)Q^{-2}(\phi_m)\mathrm{e}^{2\phi(x,t)})(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)^2\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\\
&\times\big(\big(((\lambda_1+\lambda_2)\cos(\phi_m)-2)\cos(\Theta^-(z_o,t)-\tfrac{3\pi}{4})-(\lambda_1-\lambda_2)\sin(\phi_m)\sin(\Theta^-(z_o,t)-\tfrac{3\pi}{4})\big)\cos(s^-)\\
&-\mathrm{i}\big(((\lambda_1+\lambda_2)\cos(\phi_m)-2)\cos(\Theta^-(z_o,t)-\tfrac{3\pi}{4})-(\lambda_1-\lambda_2)\sin(\phi_m)\sin(\Theta^-(z_o,t)-\tfrac{3\pi}{4})\big)\sin(s^-)\big)\,,
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{d}_m^1=&\pm\frac{2\lambda_1\sin(\phi_m)|\gamma_m|\sqrt{\nu(\lambda_1)}\,P^{-1}(\phi_m,\phi_k)Q^{-1}(\phi_m)\mathrm{e}^{\phi(x,t)}}{(1-|\gamma_m|^2P^{-2}(\phi_m,\phi_k)Q^{-2}(\phi_m)\mathrm{e}^{2\phi(x,t)})(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\\
&\times\big(2\cos(s^-)\cos(\Theta^-(z_o,t)-\tfrac{3\pi}{4})-(\lambda_1+\lambda_2)\cos(\phi_m+s^-)\cos(\Theta^-(z_o,t)-\tfrac{3\pi}{4})\\
&+(\lambda_1-\lambda_2)\sin(\phi_m+s^-)\sin(\Theta^-(z_o,t)-\tfrac{3\pi}{4})+2\mathrm{i}\sin(s^-)\cos(\Theta^-(z_o,t)-\tfrac{3\pi}{4})\\
&-\mathrm{i}(\lambda_1+\lambda_2)\sin(\phi_m+s^-)\cos(\Theta^-(z_o,t)-\tfrac{3\pi}{4})-\mathrm{i}(\lambda_1-\lambda_2)\cos(\phi_m+s^-)\sin(\Theta^-(z_o,t)-\tfrac{3\pi}{4})\big)\,,
\end{aligned}
$$

where s^- is given in Theorem 2.2.1*, Eq.* (11)*.*

The analogue of Proposition 4.5 is

Proposition A.1.6. *As $t\to-\infty$ and $x\to+\infty$ such that $z_o:=x/t<-2$ and $(x,t)\in\urcorner_m$, $m\in\{1,2,\ldots,N\}$,*

$$
\begin{aligned}
u(x,t)=&\,\mathrm{i}\left((\widetilde{\Delta}_o)_{12}+\widetilde{a}_3^-+\mathfrak{b}_m+\overline{\mathfrak{c}_m}+\frac{\sqrt{\nu(\lambda_1)}\,\mathrm{e}^{\frac{\mathrm{i}\Xi^-(0)}{2}}\lambda_1^{-2\mathrm{i}\nu(\lambda_1)}}{\sqrt{|t|(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\left(\lambda_1\mathrm{e}^{\mathrm{i}(\Theta^-(z_o,t)-\frac{3\pi}{4})}+\lambda_2\mathrm{e}^{-\mathrm{i}(\Theta^-(z_o,t)-\frac{3\pi}{4})}\right)\right)\\
&+\mathcal{O}\left(\frac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\frac{\ln|t|}{t}\right),
\end{aligned}
\tag{82}
$$

$$
\begin{aligned}
\int_{+\infty}^x(|u(x',t)|^2-1)\,\mathrm{d}x'=&-\mathrm{i}\left((\widetilde{\Delta}_o)_{11}+\widetilde{a}_1^--\widetilde{a}_2^-+\mathfrak{a}_m+\overline{\mathfrak{d}_m}+2\mathrm{i}\sum_{k=1}^{m-1}\sin(\phi_k)\right.\\
&\left.+\,\mathrm{i}\left(\int_0^{\lambda_2}+\int_{\lambda_1}^{+\infty}\right)\ln(1-|r(\mu)|^2)\,\frac{\mathrm{d}\mu}{2\pi}\right)+\mathcal{O}\left(\frac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\frac{\ln|t|}{t}\right),
\end{aligned}
\tag{83}
$$

$$
\int_{-\infty}^x(|u(x',t)|^2-1)\,\mathrm{d}x'=\int_{+\infty}^x(|u(x',t)|^2-1)\,\mathrm{d}x'-2\sum_{n=1}^N\sin(\phi_n)-\int_{-\infty}^{+\infty}\ln(1-|r(\mu)|^2)\,\frac{\mathrm{d}\mu}{2\pi}.
\tag{84}
$$

The analogue of Proposition 4.6 is

Proposition A.1.7. *As $t\to-\infty$ and $x\to+\infty$ such that $z_o:=x/t<-2$ and $(x,t)\in\urcorner_m$, $m\in\{1,2,\ldots,N\}$, for $\theta_{\gamma_m}=\pm\pi/2$,*

$$
\begin{aligned}
(\widetilde{\Delta}_o)_{11}=&\,\mathrm{i}\Bigg(\pm\frac{2\sin(\phi_m)|\gamma_m|P^{-1}(\phi_m,\phi_k)Q^{-1}(\phi_m)\mathrm{e}^{\phi(x,t)}}{(1-|\gamma_m|^2P^{-2}(\phi_m,\phi_k)Q^{-2}(\phi_m)\mathrm{e}^{2\phi(x,t)})}+\frac{\sqrt{\nu(\lambda_1)}}{\sqrt{|t|(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\big(-2\cos(\Theta^-(z_o,t)-\tfrac{3\pi}{4})\\
&\times\cos(s^-)+\frac{4\sin(\phi_m)|\gamma_m|^2P^{-2}(\phi_m,\phi_k)Q^{-2}(\phi_m)\sin(s^--\phi_m)\cos(\Theta^-(z_o,t)-\frac{3\pi}{4})\mathrm{e}^{2\phi(x,t)}}{(1-|\gamma_m|^2P^{-2}(\phi_m,\phi_k)Q^{-2}(\phi_m)\mathrm{e}^{2\phi(x,t)})}\\
&+\frac{8\lambda_1^2\sin^2(\phi_m)|\gamma_m|^2P^{-2}(\phi_m,\phi_k)Q^{-2}(\phi_m)(1+|\gamma_m|^2P^{-2}(\phi_m,\phi_k)Q^{-2}(\phi_m)\mathrm{e}^{2\phi(x,t)})\cos(s^-)\mathrm{e}^{2\phi(x,t)}}{(1-|\gamma_m|^2P^{-2}(\phi_m,\phi_k)Q^{-2}(\phi_m)\mathrm{e}^{2\phi(x,t)})^2(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)^2}\\
&\times\big(((\lambda_1+\lambda_2)\cos(\phi_m)-2)\cos(\Theta^-(z_o,t)-\tfrac{3\pi}{4})-(\lambda_1-\lambda_2)\sin(\phi_m)\sin(\Theta^-(z_o,t)-\tfrac{3\pi}{4})\big)\\
&+\frac{4\lambda_1\sin(\phi_m)\cos(\phi_m)|\gamma_m|^2P^{-2}(\phi_m,\phi_k)Q^{-2}(\phi_m)\mathrm{e}^{2\phi(x,t)}}{(1-|\gamma_m|^2P^{-2}(\phi_m,\phi_k)Q^{-2}(\phi_m)\mathrm{e}^{2\phi(x,t)})(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)}\big(2\cos(\Theta^-(z_o,t)-\tfrac{3\pi}{4})\sin(s^--\phi_m)\\
&-\,(\lambda_1+\lambda_2)\sin(s^-)\cos(\Theta^-(z_o,t)-\tfrac{3\pi}{4})-(\lambda_1-\lambda_2)\cos(s^-)\sin(\Theta^-(z_o,t)-\tfrac{3\pi}{4})\big)\big)\Bigg)\\
&+\mathcal{O}\left(\frac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\frac{\ln|t|}{t}\right),
\end{aligned}
$$

$$
\begin{aligned}
(\widetilde{\Delta}_o)_{12} =& -\mathrm{i}\mathrm{e}^{-\mathrm{i}(\theta^{-}(1)+s^{-})}+\frac{2\sin(\phi_m)|\gamma_m|^2P^{-2}(\phi_m,\phi_k)Q^{-2}(\phi_m)\mathrm{e}^{-\mathrm{i}(\theta^{-}(1)+\phi_m+s^{-})+2\phi(x,t)}}{(1-|\gamma_m|^2P^{-2}(\phi_m,\phi_k)Q^{-2}(\phi_m)\mathrm{e}^{2\phi(x,t)})}\\
&+\frac{1}{\sqrt{t}}\Bigg(\frac{2\mathrm{i}\mathrm{Im}(\widetilde{a}_1-\widetilde{a}_2)\sin(\phi_m)|\gamma_m|^2P^{-2}(\phi_m,\phi_k)Q^{-2}(\phi_m)\mathrm{e}^{-\mathrm{i}(\theta^{-}(1)+\phi_m+s^{-})+2\phi(x,t)}}{(1-|\gamma_m|^2P^{-2}(\phi_m,\phi_k)Q^{-2}(\phi_m)\mathrm{e}^{2\phi(x,t)})}\\
&\pm\frac{4\mathrm{i}\sin(\phi_m)|\gamma_m|P^{-1}(\phi_m,\phi_k)Q^{-1}(\phi_m)\sqrt{\nu(\lambda_1)}\,\mathrm{e}^{-\mathrm{i}(\theta^{-}(1)+2s^{-})+\phi(x,t)}\cos(\Theta^{-}(z_o,t)-\frac{3\pi}{4})}{(1-|\gamma_m|^2P^{-2}(\phi_m,\phi_k)Q^{-2}(\phi_m)\mathrm{e}^{2\phi(x,t)})\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\\
&\mp\frac{16\mathrm{i}\lambda_1^2\sin^2(\phi_m)|\gamma_m|^3P^{-3}(\phi_m,\phi_k)Q^{-3}(\phi_m)\sqrt{\nu(\lambda_1)}\,\mathrm{e}^{-\mathrm{i}(\theta^{-}(1)+s^{-})+3\phi(x,t)}\cos(s^{-})}{(1-|\gamma_m|^2P^{-2}(\phi_m,\phi_k)Q^{-2}(\phi_m)\mathrm{e}^{2\phi(x,t)})^2(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)^2\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\\
&\times\big(((\lambda_1+\lambda_2)\cos(\phi_m)-2)\sin(\phi_m)\cos(\Theta^{-}(z_o,t)-\tfrac{3\pi}{4})-(\lambda_1-\lambda_2)\sin^2(\phi_m)\sin(\Theta^{-}(z_o,t)-\tfrac{3\pi}{4})\\
&+\mathrm{i}(((\lambda_1+\lambda_2)\cos(\phi_m)-2)\cos(\phi_m)\cos(\Theta^{-}(z_o,t)-\tfrac{3\pi}{4})-(\lambda_1-\lambda_2)\sin(\phi_m)\cos(\phi_m)\\
&\times\sin(\Theta^{-}(z_o,t)-\tfrac{3\pi}{4})\big)+\mathrm{Im}(\widetilde{a}_1-\widetilde{a}_2)\mathrm{e}^{-\mathrm{i}(\theta^{-}(1)+s^{-})}\\
&-\frac{4\lambda_1\sin(\phi_m)|\gamma_m|P^{-1}(\phi_m,\phi_k)Q^{-1}(\phi_m)\sqrt{\nu(\lambda_1)}\,\mathrm{e}^{-\mathrm{i}(\theta^{-}(1)+s^{-})+\phi(x,t)}}{(1-|\gamma_m|^2P^{-2}(\phi_m,\phi_k)Q^{-2}(\phi_m)\mathrm{e}^{2\phi(x,t)})(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\\
&\times\big(\mp2\sin(s^{-}-\phi_m)\cos(\Theta^{-}(z_o,t)-\tfrac{3\pi}{4})\pm(\lambda_1+\lambda_2)\sin(s^{-})\cos(\Theta^{-}(z_o,t)-\tfrac{3\pi}{4})\\
&\pm(\lambda_1-\lambda_2)\cos(s^{-})\sin(\Theta^{-}(z_o,t)-\tfrac{3\pi}{4})\big)\Bigg)+\mathcal{O}\Big(\tfrac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\tfrac{\ln|t|}{t}\Big),\\
\mathrm{Im}(\widetilde{a}_1-\widetilde{a}_2) =& \pm\frac{\sqrt{\nu(\lambda_1)}\sin(s^{-})\cos(\Theta^{-}(z_o,t)-\frac{3\pi}{4})(1-|\gamma_m|^2P^{-2}(\phi_m,\phi_k)Q^{-2}(\phi_m)\mathrm{e}^{2\phi(x,t)})}{\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}\sin(\phi_m)|\gamma_m|P^{-1}(\phi_m,\phi_k)Q^{-1}(\phi_m)\mathrm{e}^{\phi(x,t)}}\\
&\pm\frac{4\lambda_1^2\sqrt{\nu(\lambda_1)}\sin(\phi_m)|\gamma_m|P^{-1}(\phi_m,\phi_k)Q^{-1}(\phi_m)\sin(s^{-})\mathrm{e}^{\phi(x,t)}}{(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)^2\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\\
&\times\big(((\lambda_1+\lambda_2)\cos(\phi_m)-2)\cos(\Theta^{-}(z_o,t)-\tfrac{3\pi}{4})-(\lambda_1-\lambda_2)\sin(\phi_m)\sin(\Theta^{-}(z_o,t)-\tfrac{3\pi}{4})\big)\\
&\pm\frac{2\lambda_1\sqrt{\nu(\lambda_1)}\cos(\phi_m)|\gamma_m|P^{-1}(\phi_m,\phi_k)Q^{-1}(\phi_m)\mathrm{e}^{\phi(x,t)}}{(\lambda_1^2-2\lambda_1\cos(\phi_m)+1)\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\big(2\cos(s^{-}-\phi_m)\cos(\Theta^{-}(z_o,t)-\tfrac{3\pi}{4})\\
&-(\lambda_1+\lambda_2)\cos(s^{-})\cos(\Theta^{-}(z_o,t)-\tfrac{3\pi}{4})+(\lambda_1-\lambda_2)\sin(s^{-})\sin(\Theta^{-}(z_o,t)-\tfrac{3\pi}{4})\big)\\
&\pm\frac{2\sqrt{\nu(\lambda_1)}|\gamma_m|P^{-1}(\phi_m,\phi_k)Q^{-1}(\phi_m)\cos(s^{-}-\phi_m)\cos(\Theta^{-}(z_o,t)-\frac{3\pi}{4})\mathrm{e}^{\phi(x,t)}}{\sqrt{(\lambda_1-\lambda_2)}\,(z_o^2+32)^{1/4}}\\
&+\mathcal{O}\Big(\tfrac{c^{\mathcal{S}}(z_o)}{(z_o^2+32)^{1/2}}\tfrac{\ln|t|}{t}\Big),
\end{aligned}
$$

$$
\mathrm{Re}(\widetilde{a}_1-\widetilde{a}_2)=\mathrm{Re}(\widetilde{a}_3)=\mathrm{Im}(\widetilde{a}_3)=0,
$$

where $\theta^{-}(\cdot)$ *is given in Theorem* 2.2.1, *Eq.* (9).

The analogue of Lemma 4.4 is

Lemma A.1.8. *As* $t\to-\infty$ *and* $x\to+\infty$ *such that* $z_o:=x/t<-2$ *and* $(x,t)\in\daleth_m$, $m\in\{1,2,\dots,N\}$, $u(x,t)$, *the solution of the Cauchy problem for the* D_fNLSE, *and* $\int_{\pm\infty}^{x}(|u(x',t)|^2-1)\,\mathrm{d}x'$ *have the leading-order asymptotic expansions (for the lower sign) stated in Theorem* 2.2.1, *Eqs.* (7)–(20).

Appendix B

In order to obtain the results of Theorems 2.2.3 and 2.2.4, the following Lemma, which is the analogue of Lemmae 4.2 and A.1.6, is requisite.

Lemma B.1.1. *Let ε be an arbitrarily fixed, sufficiently small positive real number, and, for $\lambda \in \widetilde{\mathfrak{J}} := \{(s_1)^{\pm 1}, (s_2)^{\pm 1}\}$, where*

$$s_1 = -\tfrac{1}{2}(a_1 - \mathrm{i}(4-a_1^2)^{1/2}) = \mathrm{e}^{\mathrm{i}\widehat{\varphi}_1}, \qquad \widehat{\varphi}_1 := \arctan\Big(\tfrac{(4-a_1^2)^{1/2}}{|a_1|}\Big) \in (0, \tfrac{\pi}{2}), \qquad a_1 < 0, \qquad |a_1| < 2,$$
$$s_2 = -\tfrac{1}{2}(a_2 - \mathrm{i}(4-a_2^2)^{1/2}) = \mathrm{e}^{\mathrm{i}\widehat{\varphi}_2}, \qquad \widehat{\varphi}_2 := -\arctan\Big(\tfrac{(4-a_2^2)^{1/2}}{|a_2|}\Big) \in (\tfrac{\pi}{2}, \pi), \qquad a_2 > 0, \qquad |a_2| < 2,$$

with a_1 and a_2 given in Theorem 2.2.1, Eq. (10), set $\mathbb{U}(\lambda;\varepsilon) := \{z;\, |z-\lambda| < \varepsilon\}$. Then, for $r(s_1) = \exp(-\mathrm{i}\varepsilon_1\pi/2)|r(s_1)|$, $\varepsilon_1 \in \{\pm 1\}$, $r(\overline{s_2}) = \exp(\mathrm{i}\varepsilon_2\pi/2)|r(\overline{s_2})|$, $\varepsilon_2 \in \{\pm 1\}$, $0 < r(s_2)\overline{r(\overline{s_2})} < 1$, and $\zeta \in \mathbb{C} \setminus \cup_{\lambda \in \widetilde{\mathfrak{J}}} \mathbb{U}(\lambda;\varepsilon)$, as $t \to +\infty$ and $x \to -\infty$ such that $z_o := x/t \in (-2,0)$, $m^c(\zeta)$ has the following asymptotics,

$$m_{11}^c(\zeta) = 1 + \mathcal{O}\left(\left(\frac{\underline{c}(z_o)}{(\zeta - s_1)} + \frac{\underline{c}(z_o)}{(\zeta - \overline{s_2})}\right)\frac{\mathrm{e}^{-4\alpha t}}{\beta t}\right),$$

$$m_{12}^c(\zeta) = \frac{\varepsilon_1 \mathrm{e}^{-\left(2a_0 t + \sin(\widehat{\varphi}_1)\int_{-\infty}^{0}\frac{\ln(1-|r(\mu)|^2)}{(\mu-\cos\widehat{\varphi}_1)^2+\sin^2\widehat{\varphi}_1}\frac{\mathrm{d}\mu}{\pi}\right)}\mathrm{e}^{-\mathrm{i}\left(\widehat{\varphi}_1 + \int_{-\infty}^{0}\frac{(\mu-\cos\widehat{\varphi}_1)\ln(1-|r(\mu)|^2)}{(\mu-\cos\widehat{\varphi}_1)^2+\sin^2\widehat{\varphi}_1}\frac{\mathrm{d}\mu}{\pi}\right)}}{2(|r(s_1)|)^{-1}(b_0 t)^{1/2}(\zeta - \overline{s_1})}$$
$$+ \frac{\varepsilon_2 \mathrm{e}^{-\left(2\widehat{a}_0 t - \sin(\widehat{\varphi}_3)\int_{-\infty}^{0}\frac{\ln(1-|r(\mu)|^2)}{(\mu-\cos\widehat{\varphi}_3)^2+\sin^2\widehat{\varphi}_3}\frac{\mathrm{d}\mu}{\pi}\right)}\mathrm{e}^{\mathrm{i}\left(\widehat{\varphi}_3 - \int_{-\infty}^{0}\frac{(\mu-\cos\widehat{\varphi}_3)\ln(1-|r(\mu)|^2)}{(\mu-\cos\widehat{\varphi}_3)^2+\sin^2\widehat{\varphi}_3}\frac{\mathrm{d}\mu}{\pi}\right)}}{2(|r(\overline{s_2})|)^{-1}(1 - r(s_2)\overline{r(\overline{s_2})})(\widehat{b}_0 t)^{1/2}(\zeta - s_2)}$$
$$+ \mathcal{O}\left(\left(\frac{\underline{c}(z_o)}{(\zeta - \overline{s_1})} + \frac{\underline{c}(z_o)}{(\zeta - s_2)}\right)\frac{\mathrm{e}^{-4\alpha t}}{\beta t}\right),$$

$$m_{21}^c(\zeta) = \frac{\varepsilon_1 \mathrm{e}^{-\left(2a_0 t + \sin(\widehat{\varphi}_1)\int_{-\infty}^{0}\frac{\ln(1-|r(\mu)|^2)}{(\mu-\cos\widehat{\varphi}_1)^2+\sin^2\widehat{\varphi}_1}\frac{\mathrm{d}\mu}{\pi}\right)}\mathrm{e}^{\mathrm{i}\left(\widehat{\varphi}_1 + \int_{-\infty}^{0}\frac{(\mu-\cos\widehat{\varphi}_1)\ln(1-|r(\mu)|^2)}{(\mu-\cos\widehat{\varphi}_1)^2+\sin^2\widehat{\varphi}_1}\frac{\mathrm{d}\mu}{\pi}\right)}}{2(|r(s_1)|)^{-1}(b_0 t)^{1/2}(\zeta - s_1)}$$
$$+ \frac{\varepsilon_2 \mathrm{e}^{-\left(2\widehat{a}_0 t - \sin(\widehat{\varphi}_3)\int_{-\infty}^{0}\frac{\ln(1-|r(\mu)|^2)}{(\mu-\cos\widehat{\varphi}_3)^2+\sin^2\widehat{\varphi}_3}\frac{\mathrm{d}\mu}{\pi}\right)}\mathrm{e}^{-\mathrm{i}\left(\widehat{\varphi}_3 - \int_{-\infty}^{0}\frac{(\mu-\cos\widehat{\varphi}_3)\ln(1-|r(\mu)|^2)}{(\mu-\cos\widehat{\varphi}_3)^2+\sin^2\widehat{\varphi}_3}\frac{\mathrm{d}\mu}{\pi}\right)}}{2(|r(\overline{s_2})|)^{-1}(1 - r(s_2)\overline{r(\overline{s_2})})(\widehat{b}_0 t)^{1/2}(\zeta - \overline{s_2})}$$
$$+ \mathcal{O}\left(\left(\frac{\underline{c}(z_o)}{(\zeta - s_1)} + \frac{\underline{c}(z_o)}{(\zeta - \overline{s_2})}\right)\frac{\mathrm{e}^{-4\alpha t}}{\beta t}\right),$$

$$m_{22}^c(\zeta) = 1 + \mathcal{O}\left(\left(\frac{\underline{c}(z_o)}{(\zeta - \overline{s_1})} + \frac{\underline{c}(z_o)}{(\zeta - s_2)}\right)\frac{\mathrm{e}^{-4\alpha t}}{\beta t}\right),$$

where

$$a_0 = \tfrac{1}{2}(z_o - a_1)(4-a_1^2)^{1/2} \quad (>0), \qquad \widehat{a}_0 = -\tfrac{1}{2}(z_o - a_2)(4-a_2^2)^{1/2} \quad (>0),$$
$$b_0 = \tfrac{1}{2}(z_o^2 + 32)^{1/2}(4-a_1^2)^{1/2} \quad (>0), \qquad \widehat{b}_0 = \tfrac{1}{2}(z_o^2 + 32)^{1/2}(4-a_2^2)^{1/2} \quad (>0),$$
$$\alpha := \min\{a_0, \widehat{a}_0\}, \qquad \beta := \min\{b_0, \widehat{b}_0\},$$

and, for $r(\overline{s_1}) = \exp(\mathrm{i}\varepsilon_1\pi/2)|r(\overline{s_1})|$, $\varepsilon_1 \in \{\pm 1\}$, $r(s_2) = \exp(-\mathrm{i}\varepsilon_2\pi/2)|r(s_2)|$, $\varepsilon_2 \in \{\pm 1\}$, $0 < r(s_1)\overline{r(\overline{s_1})} < 1$, and $\zeta \in \mathbb{C} \setminus \cup_{\lambda \in \widetilde{\mathfrak{J}}} \mathbb{U}(\lambda;\varepsilon)$, as $t \to -\infty$ and $x \to +\infty$ such that $z_o \in (-2,0)$,

$$m_{11}^c(\zeta) = 1 + \mathcal{O}\left(\left(\frac{\underline{c}(z_o)}{(\zeta - \overline{s_1})} + \frac{\underline{c}(z_o)}{(\zeta - s_2)}\right)\frac{\mathrm{e}^{-4\alpha|t|}}{\beta t}\right),$$

$$m_{12}^c(\zeta) = -\frac{\varepsilon_1 \mathrm{e}^{-\left(2a_0|t| - \sin(\widehat{\varphi}_1)\int_{0}^{+\infty}\frac{\ln(1-|r(\mu)|^2)}{(\mu-\cos\widehat{\varphi}_1)^2+\sin^2\widehat{\varphi}_1}\frac{\mathrm{d}\mu}{\pi}\right)}\mathrm{e}^{\mathrm{i}\left(\widehat{\varphi}_1 - \int_{0}^{+\infty}\frac{(\mu-\cos\widehat{\varphi}_1)\ln(1-|r(\mu)|^2)}{(\mu-\cos\widehat{\varphi}_1)^2+\sin^2\widehat{\varphi}_1}\frac{\mathrm{d}\mu}{\pi}\right)}}{2(|r(\overline{s_1})|)^{-1}(1 - r(s_1)\overline{r(\overline{s_1})})(b_0|t|)^{1/2}(\zeta - s_1)}$$
$$- \frac{\varepsilon_2 \mathrm{e}^{-\left(2\widehat{a}_0|t| + \sin(\widehat{\varphi}_3)\int_{0}^{+\infty}\frac{\ln(1-|r(\mu)|^2)}{(\mu-\cos\widehat{\varphi}_3)^2+\sin^2\widehat{\varphi}_3}\frac{\mathrm{d}\mu}{\pi}\right)}\mathrm{e}^{-\mathrm{i}\left(\widehat{\varphi}_3 + \int_{0}^{+\infty}\frac{(\mu-\cos\widehat{\varphi}_3)\ln(1-|r(\mu)|^2)}{(\mu-\cos\widehat{\varphi}_3)^2+\sin^2\widehat{\varphi}_3}\frac{\mathrm{d}\mu}{\pi}\right)}}{2(|r(s_2)|)^{-1}(\widehat{b}_0|t|)^{1/2}(\zeta - \overline{s_2})}$$
$$+ \mathcal{O}\left(\left(\frac{\underline{c}(z_o)}{(\zeta - s_1)} + \frac{\underline{c}(z_o)}{(\zeta - \overline{s_2})}\right)\frac{\mathrm{e}^{-4\alpha|t|}}{\beta t}\right),$$

$$
\begin{aligned}
m_{21}^{c}(\zeta)=&-\frac{\varepsilon_1 \mathrm{e}^{-\left(2a_0|t|-\sin(\widehat{\varphi}_1)\int_0^{+\infty}\frac{\ln(1-|r(\mu)|^2)}{(\mu-\cos\widehat{\varphi}_1)^2+\sin^2\widehat{\varphi}_1}\frac{\mathrm{d}\mu}{\pi}\right)}\mathrm{e}^{-\mathrm{i}\left(\widehat{\varphi}_1-\int_0^{+\infty}\frac{(\mu-\cos\widehat{\varphi}_1)\ln(1-|r(\mu)|^2)}{(\mu-\cos\widehat{\varphi}_1)^2+\sin^2\widehat{\varphi}_1}\frac{\mathrm{d}\mu}{\pi}\right)}}{2(|r(\overline{s_1})|)^{-1}(1-r(s_1)\overline{r(\overline{s_1})})(b_0|t|)^{1/2}(\zeta-\overline{s_1})} \\
&-\frac{\varepsilon_2 \mathrm{e}^{-\left(2\widehat{a}_0|t|+\sin(\widehat{\varphi}_3)\int_0^{+\infty}\frac{\ln(1-|r(\mu)|^2)}{(\mu-\cos\widehat{\varphi}_3)^2+\sin^2\widehat{\varphi}_3}\frac{\mathrm{d}\mu}{\pi}\right)}\mathrm{e}^{\mathrm{i}\left(\widehat{\varphi}_3+\int_0^{+\infty}\frac{(\mu-\cos\widehat{\varphi}_3)\ln(1-|r(\mu)|^2)}{(\mu-\cos\widehat{\varphi}_3)^2+\sin^2\widehat{\varphi}_3}\frac{\mathrm{d}\mu}{\pi}\right)}}{2(|r(s_2)|)^{-1}(\widehat{b}_0|t|)^{1/2}(\zeta-s_2)} \\
&+\mathcal{O}\left(\left(\frac{\underline{c}(z_o)}{(\zeta-\overline{s_1})}+\frac{\underline{c}(z_o)}{(\zeta-s_2)}\right)\frac{\mathrm{e}^{-4\alpha|t|}}{\beta t}\right), \\
m_{22}^{c}(\zeta)=&\,1+\mathcal{O}\left(\left(\frac{\underline{c}(z_o)}{(\zeta-s_1)}+\frac{\underline{c}(z_o)}{(\zeta-\overline{s_2})}\right)\frac{\mathrm{e}^{-4\alpha|t|}}{\beta t}\right),
\end{aligned}
$$

where $\sup_{\zeta\in\mathbb{C}\setminus\cup_{\lambda\in\tilde{\mathfrak{J}}}\mathbb{U}(\lambda;\varepsilon)}|(\zeta-(s_n)^{\pm 1})^{-1}|<\infty,$ *and* $m^c(\zeta)=\sigma_1\overline{m^c(\overline{\zeta})}\,\sigma_1.$

Appendix C. Matrix Riemann-Hilbert Theory in the L^2 Sobolev Space

In this Appendix, the theoretical foundation for this paper is presented. Beginning from the Lax-pair isospectral deformation formulation for a completely integrable NLEE, in the sense of the ISM, a succinct review of several basic and key facts from the 2×2 matrix RH factorisation theory on unbounded self-intersecting contours is presented: for complete details and proofs, see [40, 44, 45, 46, 47, 50]. For simplicity, one begins with the solitonless sector, $\sigma_d \equiv \emptyset$, leading to the so-called "regular" RHP: inclusion of the (non-empty and finitely denumerable) discrete spectrum, σ_d, is known as the "singular" RHP, and is discussed below Theorem C.1.4.

For a completely integrable system of NLEEs, in the sense of the ISM, write the spatial part of the associated Lax pair (see, for example, Proposition 2.1.1) as $\partial_x \widetilde{\Psi}(x,t;\lambda) = (\widetilde{J}(\lambda) + \widetilde{R}(x,t;\lambda))\widetilde{\Psi}(x,t;\lambda)$, where $(x,t) \in \mathbb{R} \times [-T,T]$, $\lambda \in \mathbb{C}$, $\widetilde{J}(\lambda) := \operatorname{diag}(z_1(\lambda), z_2(\lambda))$ is rational with distinct entries, and $\widetilde{R}(x,t;\lambda)$ is off-diagonal. The orders of the poles of $\widetilde{J}(\lambda)$ and $\widetilde{R}(x,t;\lambda)$ must satisfy the following requirements (denote by $\mathrm{P}_{\widetilde{J}}$ the set of poles of $\widetilde{J}(\lambda)$, and let $k(\lambda')$ denote the order of the pole of $\lambda' \in \mathrm{P}_{\widetilde{J}}$): (1) every pole of $\widetilde{R}(x,t;\lambda)$ is a pole of $\widetilde{J}(\lambda)$; (2) if ∞ is a pole of $\widetilde{J}(\lambda)$ of order $k(\infty)$, then it is a pole of $\widetilde{R}(x,t;\lambda)$ of order not greater than $k(\infty)-1$; and (3) if λ' is a finite pole of $\widetilde{J}(\lambda)$ of order $k(\lambda')$, then it is a pole of $\widetilde{R}(x,t;\lambda)$ of order not greater than $k(\lambda')$. Hence, one has the following representations for $\widetilde{J}(\lambda)$ and $\widetilde{R}(x,t;\lambda)$: (1)

$$\widetilde{J}(\lambda) = \sum_{\lambda' \in \mathrm{P}_{\widetilde{J}} \setminus \{\infty\}} \sum_{j=1}^{k(\lambda')} \widetilde{J}_{\lambda',j}(\lambda-\lambda')^{-j} + \sum_{l=0}^{k(\infty)} \widetilde{J}_{\infty,l}\lambda^l,$$

where $\widetilde{J}_{\lambda',j}$ and $\widetilde{J}_{\infty,l}$ are $\mathrm{M}_2(\mathbb{C})$-valued, diagonal matrices with distinct elements; and (2)

$$\widetilde{R}(x,t;\lambda) = \sum_{\lambda' \in \mathrm{P}_{\widetilde{J}} \setminus \{\infty\}} \sum_{j=1}^{k(\lambda')} r_{\lambda',j}(x,t)(\lambda-\lambda')^{-j} + \sum_{l=0}^{k(\infty)-1} r_{\infty,l}(x,t)\lambda^l.$$

Remark C.1.1. Hereafter, for economy of notation, all explicit x,t dependencies are suppressed.

Denote by $\widetilde{\Lambda}$ the closure of $\{\lambda \in \mathbb{C};\ \mathrm{Re}(z_1(\lambda)-z_2(\lambda))=0\}$. Decompose $\widetilde{\Lambda}$ into a finite union of piecewise smooth, simple, closed curves, $\widetilde{\Lambda} := \cup_{l \in L}\widetilde{\Lambda}_l$ ($\operatorname{card}(L) < \infty$). Denote by ϖ the set of all self-intersections of $\widetilde{\Lambda}$, $\varpi := \{\lambda;\ \widetilde{\Lambda}_l \cap \widetilde{\Lambda}_m \neq \emptyset,\ l \neq m \in \{1,2,\dots,\operatorname{card}(L)\}\}$ (it is assumed throughout that $\operatorname{card}(\varpi) < \infty$). Divide the complement of $\widetilde{\Lambda}$ into two disjoint open subsets of $\mathbb{C}$, $\mathbf{\Omega}^+$ and $\mathbf{\Omega}^-$, each of which have finitely many components, $\mathbf{\Omega}^{\pm} := \cup_{l^{\pm} \in L^{\pm}}\mathbf{\Omega}^{\pm}_{l^{\pm}}$ ($\operatorname{card}(L^{\pm}) < \infty$), such that $\widetilde{\Lambda}$ admits an orientation so that it can be viewed either as a positively (counter-clockwise) oriented boundary, $\widetilde{\Lambda}^+$, for $\mathbf{\Omega}^+$, or as a negatively (clockwise) oriented boundary, $\widetilde{\Lambda}^-$, for $\mathbf{\Omega}^-$; moreover, for each component $\mathbf{\Omega}^{\pm}_{l^{\pm}}$, $\partial\mathbf{\Omega}^{\pm}_{l^{\pm}}$ has no self-intersections.

Definition C.1.1. *For an $\mathrm{M}_2(\mathbb{C})$-valued function, $f(\lambda)$, say, denote by $f_{\pm}(\lambda)$, respectively, the non-tangential limits, if they exist, of $f(\lambda)$ taken from $\mathbf{\Omega}^{\pm}$. For $f(\lambda)\colon \widetilde{\Lambda} \to \mathrm{M}_2(\mathbb{C})$, define $f^{(0)}(\lambda) := f(\lambda)$, and, for $k \in \mathbb{Z}_{\geqslant 1}$, $f^{(j)}(\lambda) := \partial_{\lambda}^j f(\lambda)$, $j \in \{1,2,\dots,k\}$. For the piecewise smooth simple closed curve $\widetilde{\Lambda} = \cup_{l \in L}\widetilde{\Lambda}_l$, and $k \in \mathbb{Z}_{\geqslant 1}$, define the $\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Lambda})$ Sobolev space $H^k(\widetilde{\Lambda}, \mathrm{M}_2(\mathbb{C}))$ as the set of all $\mathrm{M}_2(\mathbb{C})$-valued functions on $\widetilde{\Lambda}$ satisfying: (1) for $l \in \{1,2,\dots,\operatorname{card}(L)\}$, $f^{(j)}\restriction_{\widetilde{\Lambda}_l}$, $j \in \{0,1,\dots,k-1\}$, exist pointwise and $\in \mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Lambda}_l)$; and (2) for $l \in \{1,2,\dots,\operatorname{card}(L)\}$, $f^{(k-1)}\restriction_{\widetilde{\Lambda}_l}$ is locally absolutely continuous and $f^{(k)}\restriction_{\widetilde{\Lambda}_l} \in \mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Lambda}_l)$. For $k=0$, denote $H^0(\widetilde{\Lambda}, \mathrm{M}_2(\mathbb{C}))$ by $\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Lambda})$. Define $H^k(\widetilde{\Lambda}^{\pm}, \mathrm{M}_2(\mathbb{C})) := \{f\colon \widetilde{\Lambda} \to \mathrm{M}_2(\mathbb{C});\ f\restriction_{\partial\mathbf{\Omega}^{\pm}_{l^{\pm}}} \in H^k(\partial\mathbf{\Omega}^{\pm}_{l^{\pm}}, \mathrm{M}_2(\mathbb{C})),\ l^{\pm} \in \{1,2,\dots,\operatorname{card}(L^{\pm})\}, k \in \mathbb{Z}_{\geqslant 1}\}$: the norm on $H^k(\widetilde{\Lambda}^{\pm}, \mathrm{M}_2(\mathbb{C}))$, $k \in \mathbb{Z}_{\geqslant 1}$, is defined as $\|f(\cdot)\|_{H^k(\widetilde{\Lambda}, \mathrm{M}_2(\mathbb{C}))} := \|f(\cdot)\|_{2,k} := (\sum_{l \in L}\sum_{j=0}^k \|f^{(j)}(\cdot)\|^2_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Lambda}_l)})^{1/2}$. With this norm, $H^k(\widetilde{\Lambda}^{\pm}, \mathrm{M}_2(\mathbb{C}))$ is a Hilbert space: for $k=0$, $\|f(\cdot)\|_{2,0} = (\sum_{l \in L}\|f(\cdot)\|^2_{\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Lambda}_l)})^{1/2}$.*

The Cauchy integral operators on $\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Lambda})$ are defined as $(C_{\pm}f)(\lambda) := \lim_{\substack{\lambda' \to \lambda \\ \lambda' \in \mathbf{\Omega}^{\pm}}} \int_{\widetilde{\Lambda}} \frac{f(z)}{(z-\lambda')}\frac{\mathrm{d}z}{2\pi \mathrm{i}}$: note that $C_+ - C_- = \mathbf{id}$, where $\mathbf{id}$ is the identity operator on $\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Lambda})$. Since $\widetilde{\Lambda} = \cup_{l \in L}\widetilde{\Lambda}_l$, where $\widetilde{\Lambda}_l$,

$l\in\{1,2,\dots,\mathrm{card}(L)\}$, are piecewise smooth and simple, the Cauchy integral operators are bounded from $\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Lambda})$ into $\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Lambda})$; moreover, the aforementioned orientation for $\widetilde{\Lambda}$, that is, $\widetilde{\Lambda}^{\pm}$, provides the Cauchy integral operators on $\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Lambda})$ with the crucial property that $\pm C_{\pm}$ are complementary projections, that is, $C_+^2=C_+$, $C_-^2=-C_-$, $C_+C_-=C_-C_+=\underline{\mathbf{0}}$, where $\underline{\mathbf{0}}$ is the null operator on $\mathcal{L}^2_{\mathrm{M}_2(\mathbb{C})}(\widetilde{\Lambda})$. Even though $C_{\pm}$ are not bounded in operator norm on $H^k(\widetilde{\Lambda},\mathrm{M}_2(\mathbb{C}))$, $C_{\pm}$ are bounded on $\oplus_{\alpha\in\{\pm\}}H^k(\widetilde{\Lambda}^{\alpha},\mathrm{M}_2(\mathbb{C}))$; moreover, injectively, $C_{\pm}\colon H^k(\widetilde{\Lambda}^{\pm},\mathrm{M}_2(\mathbb{C}))\to H^k(\widetilde{\Lambda}^{\pm},\mathrm{M}_2(\mathbb{C}))$, and $C_{\pm}\colon H^k(\widetilde{\Lambda}^{\mp},\mathrm{M}_2(\mathbb{C}))\to\widetilde{H}^k(\widetilde{\Lambda},\mathrm{M}_2(\mathbb{C})):=\cap_{\alpha\in\{\pm\}}H^k(\widetilde{\Lambda}^{\alpha},\mathrm{M}_2(\mathbb{C}))$. Since, in the ISM, $\widetilde{\Lambda}$ is (usually) unbounded, the function $f\restriction_{\widetilde{\Lambda}^{\pm}}=\mathrm{I}\notin H^k(\widetilde{\Lambda}^{\pm},\mathrm{M}_2(\mathbb{C}))$, $k\in\mathbb{Z}_{\geqslant 0}$; hence, for $D\in\{\widetilde{\Lambda},\widetilde{\Lambda}^{\pm}\}$, embed $H^k(D,\mathrm{M}_2(\mathbb{C}))$, $k\in\mathbb{Z}_{\geqslant 0}$, into a larger Hilbert space $H^k_I(D,\mathrm{M}_2(\mathbb{C}))$ consisting of $\mathrm{M}_2(\mathbb{C})$-valued functions $f(\lambda)$ on $\widetilde{\Lambda}\cup(\cup_{\alpha\in\{\pm\}}\mathbf{\Omega}^{\alpha})$ with the limit $f(\infty)$ at ∞ such that $f(\lambda)-f(\infty)\in H^k(D,\mathrm{M}_2(\mathbb{C}))$, with the norm defined by $\|f(\cdot)\|_{H^k_I(D,\mathrm{M}_2(\mathbb{C}))}:=\|f(\cdot)\|_{I,2,k}:=(|f(\infty)|^2+\|f(\cdot)-f(\infty)\|^2_{2,k})^{1/2}$. $H^k_I(D,\mathrm{M}_2(\mathbb{C}))$, $k\in\mathbb{Z}_{\geqslant 0}$, is isomorphic to the Hilbert space direct sum of $\mathrm{M}_2(\mathbb{C})$ and $H^k(D,\mathrm{M}_2(\mathbb{C}))$ ($H^k_I(D,\mathrm{M}_2(\mathbb{C}))\approx\mathrm{M}_2(\mathbb{C})\oplus H^k(D,\mathrm{M}_2(\mathbb{C}))$).

Define: (1) $GH^k_I(\widetilde{\Lambda}^{\pm},\mathrm{M}_2(\mathbb{C})):=\{f(\lambda)\in H^k_I(\widetilde{\Lambda}^{\pm},\mathrm{M}_2(\mathbb{C}));\ \det(f(\lambda))\not\equiv 0\}$; and (2) $SH^k_I(\widetilde{\Lambda}^{\pm},\mathrm{M}_2(\mathbb{C})):=\{f(\lambda)\in H^k_I(\widetilde{\Lambda}^{\pm},\mathrm{M}_2(\mathbb{C}));\ \det(f(\lambda))=1\}$. If $\chi^c_{\pm}(\lambda)-\chi^c_{\pm}(\infty)\in\mathrm{ran}\,C_{\pm}$ ($\subset H^k(\widetilde{\Lambda}^{\pm},\mathrm{M}_2(\mathbb{C}))$), where $\chi^c_{\pm}(\infty):=\lim_{\substack{\lambda\to\infty\\ \lambda\in\mathbf{\Omega}^{\pm}}}\chi^c(\lambda)$, denote by $\chi^c(\lambda)$ the sectionally holomorphic function on $\cup_{\alpha\in\{\pm\}}\mathbf{\Omega}^{\alpha}$ with boundary values $\chi^c_{\pm}(\lambda)$. Define: (1) $\mathcal{H}^k(\mathbb{C}\setminus\widetilde{\Lambda},\mathrm{M}_2(\mathbb{C})):=\{\chi^c(\lambda);\ \chi^c_{\pm}(\lambda)-\chi^c_{\pm}(\infty)\in\mathrm{ran}\,C_{\pm}\}$; (2) $\mathcal{GH}^k(\mathbb{C}\setminus\widetilde{\Lambda},\mathrm{M}_2(\mathbb{C})):=\{\chi^c(\lambda)\in\mathcal{H}^k(\mathbb{C}\setminus\widetilde{\Lambda},\mathrm{M}_2(\mathbb{C}));\ \det(\chi^c(\lambda))\not\equiv 0\}$; and (3) $\mathcal{SH}^k(\mathbb{C}\setminus\widetilde{\Lambda},\mathrm{M}_2(\mathbb{C})):=\{\chi^c(\lambda)\in\mathcal{H}^k(\mathbb{C}\setminus\widetilde{\Lambda},\mathrm{M}_2(\mathbb{C}));\ \det(\chi^c(\lambda))=1\}$.

Theorem C.1.1. *Every* $v(\lambda)\in GH^k_I(\widetilde{\Lambda}^-,\mathrm{M}_2(\mathbb{C}))*GH^k_I(\widetilde{\Lambda}^+,\mathrm{M}_2(\mathbb{C}))$ ($A*B:=\{xy;\ x\in A,\ y\in B\}$), $\lambda\in\widetilde{\Lambda}$, *admits an* RH *factorisation,* $v(\lambda)=(\chi^c_-(\lambda))^{-1}\blacklozenge(\lambda)\chi^c_+(\lambda)$, *where* $\blacklozenge(\lambda):=\mathrm{diag}\Big((\frac{\lambda-\lambda_+}{\lambda-\lambda_-})^{k_1},(\frac{\lambda-\lambda_+}{\lambda-\lambda_-})^{k_2}\Big)$, $\lambda_{\pm}\in\mathbf{\Omega}^{\pm}$, *and* $\chi^c(\lambda)\in\mathcal{GH}^k(\mathbb{C}\setminus\widetilde{\Lambda},\mathrm{M}_2(\mathbb{C}))$ *(*k_i, $i\in\{1,2\}$*, are called the partial indices (uniquely determined by* $v(\cdot)$ *up to a permutation) of* $v(\lambda)$*); moreover, if* $\det(v(\lambda))=1$, $\chi^c(\lambda)$ *can be chosen to be in* $\mathcal{SH}^k(\mathbb{C}\setminus\widetilde{\Lambda},\mathrm{M}_2(\mathbb{C}))$, *and* $\sum_{j=1}^2k_j=0$. *The matrix* $\chi^c(\lambda)\in\widetilde{H}^j_I(\widetilde{\Lambda},\mathrm{M}_2(\mathbb{C}))$, *for some* $j\in\{0,1,\dots,k\}$, $k\in\mathbb{Z}_{\geqslant 1}$, *is said to be a solution of the* RH *factorisation problem of* $v(\lambda)$ *if* $\chi^c_{\pm}(\lambda)-\chi^c_{\pm}(\infty)\in\mathrm{ran}\,C_{\pm}\subset H^k(\widetilde{\Lambda}^{\pm},\mathrm{M}_2(\mathbb{C}))$. *When* $v(\infty)=\mathrm{I}$ *and* $\blacklozenge(\lambda)=\mathrm{I}$, $\chi^c_{\pm}(\lambda)$ *can be uniquely determined by letting* $\chi^c_{\pm}(\infty)=\mathrm{I}$ *(canonical normalisation), in which case,* $\chi^c_{\pm}(\lambda)$, *or* $\chi^c(\lambda)$ ($\chi^c(\infty)=\mathrm{I}$), *is called the fundamental solution of the* RHP *of* $v(\lambda)$. *For the* ISM, $v(\infty)=\mathrm{I}$. *Conversely, if* $v(\lambda)$ *admits a factorisation* $v(\lambda)=(\chi^c_-(\lambda))^{-1}\blacklozenge(\lambda)\chi^c_+(\lambda)$, *then* $v(\lambda)\in GH^k_I(\widetilde{\Lambda}^-,\mathrm{M}_2(\mathbb{C}))*GH^k_I(\widetilde{\Lambda}^+,\mathrm{M}_2(\mathbb{C}))$.

Proposition C.1.1. $\mathrm{tr}(\widetilde{R}(\lambda))=0\Rightarrow\det(\chi^c(\lambda))=\mathrm{const.}$.

Definition C.1.2. *A linear operator* $\mathfrak{L}$ *on* $H^k_I(D,\mathrm{M}_2(\mathbb{C}))$ *is Fredholm if:* (1) *the complement of* $\mathrm{ran}\,\mathfrak{L}$ *is open in* $H^k_I(D;\mathrm{M}_2(\mathbb{C}))$; *and* (2) $\dim\ker(\mathfrak{L})$ *and* $\dim\mathrm{coker}(\mathfrak{L})$ *are finite. For* $\mathfrak{L}$ *linear and Fredholm,* $i(\mathfrak{L}):=\dim\ker(\mathfrak{L})-\dim\mathrm{coker}(\mathfrak{L})$ *is called the (Fredholm) index of* $\mathfrak{L}$.

Theorem C.1.2. *Let* $k\in\mathbb{Z}_{\geqslant 1}$. *If* $v(\lambda)$ *in Theorem* C.1.1 *can be represented as the following (algebraic) block triangular factorisation,* $v(\lambda):=(\mathrm{I}-w^-(\lambda))^{-1}(\mathrm{I}+w^+(\lambda))$, $\lambda\in\widetilde{\Lambda}$, *where* $w^{\pm}(\lambda)\in H^k(\widetilde{\Lambda}^{\pm},\mathrm{M}_2(\mathbb{C}))$, $\mathrm{I}\pm w^{\pm}(\lambda)\in GH^k_I(\widetilde{\Lambda}^{\pm},\mathrm{M}_2(\mathbb{C}))$, *and* $w^{\pm}(\lambda)$ *are nilpotent, with degree of nilpotency 2, and if, as a linear operator on* $\widetilde{H}^k_I(\widetilde{\Lambda},\mathrm{M}_2(\mathbb{C})):=\cap_{\alpha\in\{\pm\}}H^k_I(\widetilde{\Lambda}^{\alpha},\mathrm{M}_2(\mathbb{C}))$, $C_w\colon\widetilde{H}^k_I(\widetilde{\Lambda},\mathrm{M}_2(\mathbb{C}))\to\widetilde{H}^k_I(\widetilde{\Lambda},\mathrm{M}_2(\mathbb{C}))$ *is defined as* ($f\in\widetilde{H}^k_I(\widetilde{\Lambda},\mathrm{M}_2(\mathbb{C}))$) $f\mapsto C_+(fw^-)+C_-(fw^+)$, *then* $\mathbf{id}-C_w$, *where* $\mathbf{id}$ *is the identity operator on* $\widetilde{H}^k_I(\widetilde{\Lambda},\mathrm{M}_2(\mathbb{C}))$, *is Fredholm, that is,* $i(\mathbf{id}-C_w)=\dim\ker(\mathbf{id}-C_w)-\dim\mathrm{coker}(\mathbf{id}-C_w)=0$, $\dim\ker(\mathbf{id}-C_w)=2\sum_{k_j>0}k_j$, *and* $\dim\mathrm{coker}(\mathbf{id}-C_w)=-2\sum_{k_j<0}k_j$, *where* k_i, $i\in\{1,2\}$, *are the partial indices of* $v(\lambda)$*; moreover,* $i(\mathbf{id}-C_w)=2\,\mathrm{ind}\det(v(\lambda))=\frac{1}{\pi}\int_{\widetilde{\Lambda}}d(\arg\det(v(\cdot)))=0$, *where* $\mathrm{ind}\det(v(\lambda))$, *the index of* $\det(v(\lambda))$, *equals* $\sum_{j=1}^2k_j$. *Define* $\chi^c_o(\lambda):=((\mathbf{id}-C_w)^{-1}\mathrm{I})(\lambda)$*: then the boundary values* $\chi^c_{\pm}(\lambda):=\chi^c_o(\lambda)(\mathrm{I}\pm w^{\pm}(\lambda))\in(\mathrm{I}+\mathrm{ran}\,C_{\pm})\cap GH^k_I(\widetilde{\Lambda}^{\pm},\mathrm{M}_2(\mathbb{C}))\subset(\mathrm{I}+H^k(\widetilde{\Lambda}^{\pm},\mathrm{M}_2(\mathbb{C})))\cap GH^k_I(\widetilde{\Lambda}^{\pm},\mathrm{M}_2(\mathbb{C}))$ *give the fundamental solution of the* RH *factorisation problem for* $v(\lambda)$.

Theorem C.1.3. *If all the partial indices of* $v(\lambda)$ *are zero* ($k_i=0$, $i\in\{1,2\}$), *then the Fredholm operator* $\mathbf{id}-C_w$ *is invertible on* $\widetilde{H}^k_I(\widetilde{\Lambda},\mathrm{M}_2(\mathbb{C}))$, *namely,* $\ker(\mathbf{id}-C_w)=\emptyset$ ($\dim\ker(\mathbf{id}-C_w)=0$).

Lemma C.1.1. *The* RHP *of* $v(\lambda):=(\mathrm{I}-w^-(\lambda))^{-1}(\mathrm{I}+w^+(\lambda))=(\chi^c_-(\lambda))^{-1}\chi^c_+(\lambda)$, $\lambda\in\widetilde{\Lambda}$, *where* $w^{\pm}(\lambda)\in H^k(\widetilde{\Lambda}^{\pm},\mathrm{M}_2(\mathbb{C}))$, *has a fundamental solution* ($\chi^c(\infty)=\mathrm{I}$, $\chi^c(\lambda)\not\equiv 0$) *only if* $\frac{1}{2\pi}\int_{\widetilde{\Lambda}}d(\arg\det(v(\cdot)))=0$.

Conversely, if $\chi^{c}(\lambda) \in \widetilde{H}_{I}^{k}(\widetilde{\Lambda}, \mathrm{M}_2(\mathbb{C}))$, $k \in \mathbb{Z}_{\geqslant 1}$, $\chi^{c}(\infty) = \mathrm{I}$ *is a solution of the* RHP *of* $v(\lambda)$ *on* $\widetilde{\Lambda}$, *and* $\frac{1}{2\pi}\int_{\widetilde{\Lambda}} d(\arg\det(v(\cdot))) = 0$, *then* $\chi^{c}(\lambda)$ *is a fundamental solution; furthermore,* $\det(v(\lambda)) = 1 \Rightarrow \det(\chi^{c}(\lambda)) = 1$.

Proposition C.1.2. *If the* RHP *of* $v(\lambda) := (\mathrm{I} - w^{-}(\lambda))^{-1}(\mathrm{I} + w^{+}(\lambda)) = (\chi_{-}^{c}(\lambda))^{-1}\chi_{+}^{c}(\lambda)$, $\lambda \in \widetilde{\Lambda}$, *where* $w^{\pm}(\lambda) \in H^{k}(\widetilde{\Lambda}^{\pm}, \mathrm{M}_2(\mathbb{C}))$, *admits a fundamental solution* $\chi^{c}(\lambda) \in \widetilde{H}_{I}^{j}(\widetilde{\Lambda}, \mathrm{M}_2(\mathbb{C}))$ *for some* $j \in \mathbb{Z}_{\geqslant 1}$, *then it is unique in* $\mathcal{L}_{I}^{2}(\widetilde{\Lambda}, \mathrm{M}_2(\mathbb{C})) := \widetilde{H}_{I}^{0}(\widetilde{\Lambda}, \mathrm{M}_2(\mathbb{C}))$.

Proposition C.1.3. *If the* RHP *of* $v(\lambda) := (\mathrm{I} - w^{-}(\lambda))^{-1}(\mathrm{I} + w^{+}(\lambda)) = (\chi_{-}^{c}(\lambda))^{-1}\chi_{+}^{c}(\lambda)$, $\lambda \in \widetilde{\Lambda}$, *where* $w^{\pm}(\lambda) \in H^{k}(\widetilde{\Lambda}^{\pm}, \mathrm{M}_2(\mathbb{C}))$, *admits a fundamental solution* $\chi^{c}(\lambda) \in \widetilde{H}_{I}^{j}(\widetilde{\Lambda}, \mathrm{M}_2(\mathbb{C}))$ *for some* $j \in \mathbb{Z}_{\geqslant 0}$, *then* $\mathbf{id} - C_{w}$ *is invertible on* $\widetilde{H}_{I}^{j'}(\widetilde{\Lambda}, \mathrm{M}_2(\mathbb{C}))$ $\forall\, j' \in \{0, 1, \ldots, k\}$, $k \in \mathbb{Z}_{\geqslant 1}$.

Proposition C.1.4. *Suppose that* $w^{\pm}(\lambda) \in H^{k}(\widetilde{\Lambda}^{\pm}, \mathrm{M}_2(\mathbb{C}))$. *If* $\mathbf{id} - C_{w}$ *is invertible on* $\widetilde{H}_{I}^{j}(\widetilde{\Lambda}, \mathrm{M}_2(\mathbb{C}))$ *for any* $j \leqslant k$, $k \in \mathbb{Z}_{\geqslant 1}$, *then it is invertible* $\forall\, j \leqslant k$.

Denote the Schwarz reflection of an $\mathrm{M}_2(\mathbb{C})$-valued function by $f^{\mathcal{S}}(\lambda) := (f(\overline{\lambda}))^{\dagger}$, where † denotes Hermitian conjugation, and, for a subset of $\mathbb{C}$, as the reflection about $\mathbb{R}$.

Theorem C.1.4. *If* $\widetilde{\Lambda}$ *is a Schwarz reflection invariant contour about* $\mathbb{R}$, $v(\lambda) \in SH_{I}^{k}(\widetilde{\Lambda}^{-}, \mathrm{M}_2(\mathbb{C})) * SH_{I}^{k}(\widetilde{\Lambda}^{+}, \mathrm{M}_2(\mathbb{C}))$, $v(\infty) = \mathrm{I}$, $v(\cdot)$ *is positive definite on* $\mathbb{R}$, $\mathrm{Re}(v(\lambda))\restriction_{\mathbb{R}} > 0$, *and* $v(\lambda)\restriction_{\widetilde{\Lambda}\setminus\mathbb{R}} = \sigma^{-1} v^{\mathcal{S}}(\lambda)\restriction_{\widetilde{\Lambda}\setminus\mathbb{R}} \cdot\sigma$, *where* σ *is a constant, invertible, finite-order matrix involution which changes the sign(s) of some (or all) of the elements of the matrix on which it (and its inverse) is multiplied, then all the partial indices of* $v(\lambda)$ *are zero,* $k_i = 0$, $i \in \{1, 2\}$. *In this case, the* RHP *for* $v(\lambda)$ *is solvable.*

The singular RHP, that is, the RH factorisation problem with isolated singularities (in this work, first-order poles), is now introduced. Let $\zeta \in \mathbb{C}$. For the remainder of this Appendix, the same symbol is used to denote an $\mathrm{M}_2(\mathbb{C})$-valued function analytic in a punctured neighbourhood of ζ and the germ (the set of equivalence classes of analytic continuations) at ζ it represents, with the algebra of all such germs denoted by $\mathcal{A}_{\zeta}$, and $S\mathcal{A}_{\zeta} := \{\varphi_{\zeta}(\lambda) \in \mathcal{A}_{\zeta};\, \det(\varphi_{\zeta}(\lambda)) = 1\}$. Let $\mathfrak{D} \subset \mathbb{C}$, with $\mathrm{card}(\mathfrak{D}) < \infty$. Set $\mathfrak{D}^{\pm} := \mathfrak{D} \cap \boldsymbol{\Omega}^{\mp}$. Define $H^{k}(\widetilde{\Lambda}^{\pm} \cup \mathfrak{D}, \mathrm{M}_2(\mathbb{C})) := H^{k}(\widetilde{\Lambda}^{\pm}, \mathrm{M}_2(\mathbb{C})) \oplus (\oplus_{\zeta \in \mathfrak{D}} \mathcal{A}_{\zeta})$. An element in $\oplus_{\alpha \in \{\pm\}} H^{k}(\widetilde{\Lambda}^{\alpha} \cup \mathfrak{D}, \mathrm{M}_2(\mathbb{C}))$ is represented either as $\varphi(\lambda) := (\varphi_{c}(\lambda), \varphi_{\zeta}(\lambda))_{\zeta \in \mathfrak{D}}$, or

$$\varphi(\lambda) := \begin{cases} \varphi_{c}(\lambda), & \lambda \in \widetilde{\Lambda}, \\ \varphi_{\zeta}(\lambda), & \lambda \approx \zeta, \quad \zeta \in \mathfrak{D}, \end{cases}$$

where $\varphi_{c}(\lambda) \in \oplus_{\alpha \in \{\pm\}} H^{k}(\widetilde{\Lambda}^{\alpha}, \mathrm{M}_2(\mathbb{C}))$, and $\varphi_{\zeta}(\lambda) \in \mathcal{A}_{\zeta}$ (in the above, the subscript c is used to connote “continuous”, while the subscript ζ (for $\zeta \in \mathfrak{D}$) is used to connote “discrete”). The Cauchy integral operators, $C_{\pm}$, are defined on $\oplus_{\alpha \in \{\pm\}} H^{k}(\widetilde{\Lambda}^{\alpha} \cup \mathfrak{D}, \mathrm{M}_2(\mathbb{C}))$ in the following sense: construct the *augmented* contour $\widetilde{\Lambda}_{\mathrm{aug}} := \widetilde{\Lambda} \cup (\cup_{\zeta \in \mathfrak{D}} \widetilde{\Lambda}_{\zeta})$, where $\widetilde{\Lambda}_{\zeta}$ are sufficiently small, mutually disjoint, and disjoint with respect to $\widetilde{\Lambda}$, disks oriented counter-clockwise (respectively, clockwise) $\forall\, \zeta \in \mathfrak{D}^{+}$ (respectively, $\forall\, \zeta \in \mathfrak{D}^{-}$). Since, with the above-given conditions on $\widetilde{\Lambda}_{\zeta}$, $\zeta \in \mathfrak{D}$, and, for each $\varphi(\lambda) \in \oplus_{\alpha \in \{\pm\}} H^{k}(\widetilde{\Lambda}^{\alpha} \cup \mathfrak{D}, \mathrm{M}_2(\mathbb{C}))$, $\varphi(\lambda)\restriction_{\lambda \in \widetilde{\Lambda}_{\mathrm{aug}}} \exists$, it represents an element in $\oplus_{\alpha \in \{\pm\}} H^{k}(\widetilde{\Lambda}_{\mathrm{aug}}^{\alpha}, \mathrm{M}_2(\mathbb{C}))$; hence, $(C_{\pm}\varphi)(\lambda)$ are defined, and $(C_{\pm}\varphi)(\lambda) \in H^{k}(\widetilde{\Lambda}_{\mathrm{aug}}^{\pm}, \mathrm{M}_2(\mathbb{C}))$. Hereafter, $(C_{\pm}\varphi)(\lambda)$ are to be understood as elements in $H^{k}(\widetilde{\Lambda}^{\pm} \cup \mathfrak{D}, \mathrm{M}_2(\mathbb{C}))$. For $\zeta \in \mathfrak{D}^{+}$, $(C_{+}\varphi)(\lambda)$ extends analytically into the disk bounded by $\widetilde{\Lambda}_{\zeta}$, and $(C_{-}\varphi)(\lambda) := (C_{+}\varphi)(\lambda) - \varphi(\lambda)$ extends analytically into the punctured disk; therefore, they represent germs in $\mathcal{A}_{\zeta}$, denoted by $f_{\zeta}^{\pm}$, respectively. Similarly, for $\zeta \in \mathfrak{D}^{-}$, $(C_{-}\varphi)(\lambda)$ extends analytically into the disk bounded by $\widetilde{\Lambda}_{\zeta}$, and $(C_{+}\varphi)(\lambda) := (C_{-}\varphi)(\lambda) + \varphi(\lambda)$ extends analytically into the punctured disk; therefore, they represent germs in $\mathcal{A}_{\zeta}$, denoted by $f_{\zeta}^{\mp}$, respectively. Write $f_{c}^{\pm} := (C_{\pm}\varphi)(\lambda)\restriction_{\lambda \in \widetilde{\Lambda}}$, and define $f^{\pm} := (f_{c}^{\pm}, f_{\zeta}^{\pm})_{\zeta \in \mathfrak{D}} \in H^{k}(\widetilde{\Lambda}^{\pm} \cup \mathfrak{D}, \mathrm{M}_2(\mathbb{C}))$. From the construction above, $C_{\pm} \colon \oplus_{\alpha \in \{\pm\}} H^{k}(\widetilde{\Lambda}^{\alpha} \cup \mathfrak{D}, \mathrm{M}_2(\mathbb{C})) \to H^{k}(\widetilde{\Lambda}^{\pm} \cup \mathfrak{D}, \mathrm{M}_2(\mathbb{C}))$, and $(C_{\pm}\varphi)(\lambda) = f^{\pm}$. In this sense, $C_{\pm}$ are called the Cauchy integral operators with singular support $\widetilde{\Lambda} \cup \mathfrak{D}$. The following notion of piecewise-holomorphic matrix-valued function has been used throughout this paper. For an $\mathrm{M}_2(\mathbb{C})$-valued function, $\Psi(\lambda)$, say, the “symbol” $\Psi(\lambda) := (\Psi_{c}(\lambda), \Psi_{\zeta}(\lambda))_{\zeta \in \mathfrak{D}}$ is said to be a piecewise-holomorphic matrix-valued function with respect to the contour $\widetilde{\Lambda} \cup \mathfrak{D}$ if $\Psi_{c}(\lambda)$ is a piecewise-holomorphic matrix-valued

function on $\boldsymbol{\Omega} \setminus \mathcal{D}$ and $\Psi_\zeta(\lambda) \in \mathcal{A}_\zeta$ is analytic at each $\zeta \in \mathcal{D}$. The boundary values $\Psi_\pm(\lambda)$, if they exist, of the (generalised) holomorphic matrix-valued function $\Psi(\lambda) := (\Psi_c(\lambda), \Psi_\zeta(\lambda))_{\zeta \in \mathcal{D}}$ are defined by

$$\Psi_+(\lambda) := \begin{cases} (\Psi_c(\lambda))_+, & \lambda \in \widetilde{\Lambda}, \\ \Psi_c(\lambda), & \lambda \approx \zeta, \quad \zeta \in \mathcal{D}^-, \\ \Psi_\zeta(\lambda), & \lambda \approx \zeta, \quad \zeta \in \mathcal{D}^+, \end{cases} \qquad \Psi_-(\lambda) := \begin{cases} (\Psi_c(\lambda))_-, & \lambda \in \widetilde{\Lambda}, \\ \Psi_c(\lambda), & \lambda \approx \zeta, \quad \zeta \in \mathcal{D}^+, \\ \Psi_\zeta(\lambda), & \lambda \approx \zeta, \quad \zeta \in \mathcal{D}^-, \end{cases} \tag{C.1}$$

where $(\Psi_c(\lambda))_\pm := \lim_{\substack{\lambda' \to \lambda \\ \lambda' \in \boldsymbol{\Omega}^\pm}} \Psi_c(\lambda')$. Define $\mathcal{H}^k(\mathbb{C} \setminus \widetilde{\Lambda} \cup \mathcal{D}, \mathrm{M}_2(\mathbb{C})) := \{\Psi(\lambda);\ \Psi_\pm(\lambda) - \Psi_\pm(\infty) \in \operatorname{ran} C_\pm\}$, and $\mathcal{SH}^k(\mathbb{C} \setminus \widetilde{\Lambda} \cup \mathcal{D}, \mathrm{M}_2(\mathbb{C})) := \{\Psi(\lambda) \in \mathcal{H}^k(\mathbb{C} \setminus \widetilde{\Lambda} \cup \mathcal{D}, \mathrm{M}_2(\mathbb{C}));\ \det(\Psi(\lambda)) = 1\}$.

Theorem C.1.5. *Every* $v(\lambda) \in SH_I^k(\widetilde{\Lambda}^- \cup \mathcal{D}, \mathrm{M}_2(\mathbb{C})) * SH_I^k(\widetilde{\Lambda}^+ \cup \mathcal{D}, \mathrm{M}_2(\mathbb{C}))$ *admits an* RH *factorisation* $v(\lambda) := (\chi_-(\lambda))^{-1} \blacklozenge(\lambda) \chi_+(\lambda)$, *where* $\chi(\lambda) \in \mathcal{SH}^k(\mathbb{C} \setminus \widetilde{\Lambda} \cup \mathcal{D}, \mathrm{M}_2(\mathbb{C}))$, $\blacklozenge(\lambda)$ *is defined in Theorem* C.1.1, *and* $\lambda_\pm \in \mathcal{D}^\pm \cup (\boldsymbol{\Omega}^\pm \setminus \mathcal{D}^\mp)$.

Theorem C.1.6. *If* $\widetilde{\Lambda} \cup \mathcal{D}$ *is Schwarz reflection invariant with respect to* $\mathbb{R}$, $v(\lambda) \in SH_I^k(\widetilde{\Lambda}^- \cup \mathcal{D}, \mathrm{M}_2(\mathbb{C})) * SH_I^k(\widetilde{\Lambda}^+ \cup \mathcal{D}, \mathrm{M}_2(\mathbb{C}))$, $v(\infty) = \mathrm{I}$, $\operatorname{Re}(v(\lambda))\restriction_{\lambda \in \mathbb{R}} > 0$, *and* $v(\lambda)\restriction_{\lambda \in (\widetilde{\Lambda} \cup \mathcal{D}) \setminus \mathbb{R}} = \sigma^{-1} v^{\mathcal{S}}(\lambda)\restriction_{\lambda \in (\widetilde{\Lambda} \cup \mathcal{D}) \setminus \mathbb{R}} \sigma$, *where* σ *is a constant, invertible, finite-order matrix involution which changes the sign(s) of some (or all) of the elements of the matrix on which it (and its inverse) is multiplied, then all the partial indices of* $v(\lambda)$ *are zero,* $k_i = 0$, $i \in \{1, 2\}$. *In this case, the* RHP *for* $v(\lambda)$ *is solvable.*

Note that, for $\mathcal{D} \equiv \emptyset$, Theorem C.1.6 reduces to Theorem C.1.4. The asymptotic analysis of the latter part of the above-given paradigm, related to the singular RHP (when $\mathcal{D} \not\equiv \emptyset$ and $\operatorname{card}(\mathcal{D}) < \infty$), is the subject of the present asymptotic study. Using the results of this subsection, the very important Lemma 2.4 of [50], and the Deift-Zhou non-linear steepest descent method [55], the (rigorous) asymptotic analysis, as $|t| \to \infty$ and $|x| \to \infty$ such that $z_o := x/t \sim \mathcal{O}(1)$ and $\in \mathbb{R} \setminus \{-2, 0, 2\}$, of the RHP for $m(\zeta)$ formulated in Lemma 2.1.2, for $\sigma_d \equiv \emptyset$, was completed in [38].

Acknowledgements

The author is very grateful to X. Zhou for the invitation to Duke University and for the opportunity to complete this work. The author is also grateful to the referees for helpful suggestions.

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